Indexers and Comovement

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Abstract
I introduce a general equilibrium model with active investors and indexers. The presence of indexers causes market segmentation, and the degree of segmentation is linked to the relative wealth of indexers in the economy. Any shock to this relative wealth generates excess comovement by inducing correlated shocks to discount rates of index stocks. The wealthier the indexers are, the greater the resulting excess comovement is. In the data, I find that S&P 500 stocks tend to comove more with other index stocks and less with non-index stocks, but this was not the case until the 1970s when indexing gained in popularity. I use passive holdings of S&P 500 stocks as a proxy for the wealth of indexers and find that changes in passive holdings are positively related to changes of excess comovement in S&P 500 stocks.

Keywords: Asset pricing, indexing, comovement, general equilibrium.

JEL Classifications: G11, G12, G14

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1 Introduction

The average S&P 500 stock comoves more with the index than with the rest of the stock market, but this was not always the case. Figure 1 illustrates how the average comovement $\beta$s of S&P 500 stocks changed over the years since the index was created in 1957. Prior to 1975, the average index stock comoved more with the rest of the market than with the index. This pattern reversed over the following 15 years. In their seminal study of stock return comovement, Barberis, Shleifer, and Wurgler (2005) show that the impact of joining the S&P 500 on comovement is stronger in the later half of their sample. They suggest this finding might be caused by the increase in indexing. I investigate this hypothesis and show, in a theoretical framework and empirically, that an increase in the level of indexing causes index stocks to comove more with other index stocks and less with non-index stocks.

![Figure 1](image-url)

**Figure 1**

Average annual comovement $\beta$s of S&P 500 stocks from 1957 to 2010 estimated using a $[+1,+12]$ months window from the bivariate regression $R_{j,t} = \alpha_j + \beta_{j,SP500}R_{SP500,t} + \beta_{j,nonSP500}R_{nonSP500,t} + u_{j,t}$. Results are for stocks in the index on the last trading day of the year, using returns from the following year.

Understanding comovement is central to the way we think about risk in finance. The most common way to adjust returns for risk in the academic literature is to obtain excess returns after controlling for comovement with the Fama-French (or Fama-French-Cahart) risk factors. As Cremers, Petajisto, and Zitzewitz (2010) show, these factors are highly correlated with

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1Details regarding the empirical methodology are presented in Section 5.
returns of popular indices. For example, correlation of the Market-Rf factor with the S&P 500 from 1986-2005 is 0.981. Correlation of the Small minus Big factor with the Russell 2000 minus the S&P 500 is 0.933. This alignment of common risk factors with popular indices, combined with an increase in index-linked comovement, causes index membership to emerge as a new risk factor. I aim to shed some light on how we should think about this new risk factor going forward by explaining how index-linked comovement arises.

It is natural to think the rise of indexing could have caused this increase in comovement of index stocks. The share of index-linked investments increased dramatically over the past 15 years, in part due to the introduction of liquid, low-fee passive exchange-traded funds (ETF). Figure 2 illustrates that the share of ETF and index funds grew from 3.25% of all US equity mutual funds in 1993 to almost 25% in 2010. Passive investments in the S&P 500 led the trend; it is estimated that by 1990 around 10% of the market cap of the index was held by indexers.

![Figure 2](http://www.icifactbook.org/)

Share of assets under management in US Equity mutual funds and exchange-traded funds that are invested passively, broken down in the share of passive mutual fund (bottom) and ETFs (top). Data is from the 2011 Investment Company Institute Fact Book. [http://www.icifactbook.org/](http://www.icifactbook.org/).

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2Source: 2011 Investment Company Institute Fact Book. [http://www.icifactbook.org/](http://www.icifactbook.org/). For the full investment landscape including pension funds, public/private funds, endowments and hedge funds, data from [French (2008)](http://www.icifactbook.org/) suggests that in 2006 at least 10% of all investments in US equities were passive. This is a lower bound since no data was available on passive investments by foreign investors and foreign holdings who held about 40% of US equities in 2006.
I first present an asset-pricing model of indexing in which comovement arises endogenously. The continuous-time economy is populated by two rational agents, an active investor and an indexer, who must allocate their wealth between a riskless bond and three risky stocks. Two of the stocks form a value-weighted index, and the indexer is constrained to invest only in the bond and those two stocks, according to index weights. The third stock is only available to the active investor, so in equilibrium she holds the full supply. Having three stocks allows me to study the comovement of one index stock with the other index stock and with the non-index stock. I obtain closed-form approximations for equilibrium prices and moments of stock returns. In the model, index stocks have lower Sharpe ratios and higher prices than non-index stocks and they comove more with other index stocks than with non-index stocks. These effects are stronger the larger the relative wealth of indexers is.

There are three sources of comovement in the model. First, the fundamental dividend streams may be correlated. Second, as in Cochrane, Longstaff, and Santa-Clara (2008) and Martin (2012), when one dividend stream gets a positive shock it becomes larger relative to the other assets (its dividend share increases). To remain diversified investors demand more of the other assets, generating correlated price returns. These two sources of comovement are present in frameworks with unconstrained agents and affect index stocks and non-index stocks alike. The third source of comovement, which I call excess comovement, is due to the indexing constraint. Since the two agents hold different portfolios, a dividend shock to any asset changes their relative wealth. This induces them to change their demand for risky stocks, which affects the price of risk. The indexer has demand only for the index stocks, so the resulting discount rate shocks for the two index stocks are positively related. The discount rate shock for the non-index stock is negatively related to those for the index stock. Furthermore, the sensitivity of the discount rate to changes in wealth is not constant; it is increasing in relative wealth of the indexer. This makes excess comovement more pronounced when the indexer is wealthier.

I test these model predictions regarding excess comovement using S&P 500 stocks. I use the amount invested passively in the index divided by the total value of the market as a proxy for the relative wealth of indexers. I call this measure passive ownership (PO). Comovement of S&P 500 stocks with other index stocks increases as passive ownership increases. Inversely, comovement with non-S&P 500 stocks decreases as passive ownership increases. This relationship is not present in the largest 250 stocks that are not part of the S&P 500, even though their size is comparable to the smallest 250 stocks in the index.

2 Related literature

In a frictionless economy with rational investors, prices reflect fundamental value so any comovement in prices reflects common changes in fundamentals or discount rates. Cochrane, Longstaff,
and Santa-Clara (2008) show that in an economy with two Lucas trees available in fixed supply, market clearing causes stock returns to be correlated even if the dividend streams are not. When one of the trees enjoys a positive shock, its share of aggregate dividend increases, lowering the other asset’s share. Lowering the share of dividend typically raises the price-dividend ratio – causing a positive return – because the asset provides more diversification benefit. However, the literature on comovement presents evidence suggesting the presence of excess comovement within index stocks.3

To explain this phenomenon, prior research developed friction-based and sentiment-based theories of comovement. Barberis and Shleifer (2003) analyze the idea of style investing (also called the category view) and its relationship to comovement. My model of indexing is based on this idea of style investing. This theory states that investors group assets into different categories based on some characteristic, such as size or book-to-market ratio, in order to simplify portfolio decisions. They then allocate funds at the style level rather than at the individual asset level, generating correlated flows to assets in the same style. Peng and Xiong (2006) present a model in which investors have scarce cognitive resources. Investors prefer to learn category-level information, and the resulting endogenous style investing causes comovement. Another theory for comovement is the habitat view, which proposes that some investors restrict their investments to a subset of available assets due to frictions such as trading costs or information availability. When these investors change their portfolio because of liquidity needs, change in risk aversion, or other idiosyncratic events, they induce correlated trading for the subset of assets they invest in. Finally, the information diffusion view of comovement proposes that comovement arises because information is incorporated at different rate for different assets. Thus, assets prices for which information is incorporated at the same rate will comove more because of time-synchronicity in trading. A related view is that when faced with new information, investors for adjust their index exposure before evaluating stocks individually.

Barberis, Shleifer, and Wurgler (2005) investigate these different views of comovement by looking at S&P 500 additions and deletions. They find evidence consistent with all three views of comovement. Focusing on stocks that should incorporate information at similar rate, Greenwood (2008) and Boyer (2011) provide strong evidence consistent with the style and habitat views of comovement. While my story is based on the category view, I do not reject the other explanations. I provide evidence suggesting that pure indexers play a role generating excess comovement in index stocks, but it is likely that other factors also contribute to this phenomenon.

Another use of indices that could be related to comovement is benchmarking in delegated

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3See, for example, Vijh (1994), Barberis, Shleifer, and Wurgler (2005) and Boyer (2011) for evidence of comovement in S&P indices or Shiller (1989) for earlier evidence of excess comovement in equities.
portfolio management. Basak and Pavlova (2012) find predictions that are similar to mine in the presence of benchmarking: discount rates are lower for index stocks, and index stocks comove more with other index stocks than with non-index stocks. However, in their model non-index stocks are not affected by the wealth of benchmarked investors. The wealth of indexers does not affect the correlation between index stocks and non-index stocks, which remains zero. Institutional investors have been associated with excess comovement early on. For example, Pindyck and Rotemberg (1993) show that excess comovement in equities is greater in stocks with high institutional ownership. Grouping institutional investors by their level of activity, Ye (2012) argues that active investors can eliminate excess comovement, but that passive investors play not role in comovement. This conclusion regarding the role of passive investors is contrary to my hypothesis, and I believe it is flawed by the way passive investors are identified. Using the Active Share measure of Cremers and Petajisto (2009), he ranks institutional investors based on the level of activity and labels as passive investors those in the lowest tercile. Under this definition, investors identified as passive could be true indexers, closet indexers or even active investors who are less active relative to other active investors. The underlying assumption is that investors in the bottom tercile – a proportion that is fixed through time – are passive, while my story is about the impact of changes in passive investing.

To some, the growth in indexing is not surprising. French (2008) makes an argument in favor of passive investment: it is cheaper than active investment and, since trading is a zero-sum game, it is not possible to beat the market on average. He argues that, compared to a world in which everyone holds the market, investors spend 0.67% of the aggregate value of the market annually searching for superior returns. Savov (2009) argues however that because of rebalancing, indexers fail to attain buy-and-hold index returns. He finds that high index fund flows forecast low index fund returns relative to active fund returns, and this can account for most of the differential alphas between the two types of funds. Pastor and Stambaugh (2012) propose an explanation for the presence of active investing, despite historical evidence of negative alpha, based on the idea of decreasing returns to scale. In their model, active investing yields positive alpha if there are not many active investors. As the weight of active investors increases, alphas decrease and can even become negative. They show that after calibration, their model can explain the rise of indexing because investors learn about the parameters of the returns function. The alternative to active investing proposed by French (2008) is to maximize diversification by tracking indices that cover the total market such as the Wilshire 5000 or the Russell 3000E. However, in practice the most popular US equity indices remain the S&P 500 and the Russell 2000, which are cap-based subsets of the total market. Therefore, passive investors

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5The idea of decreasing returns to scale was made popular by Berk and Green (2004). The difference is that Pastor and Stambaugh (2012) apply the idea to aggregate active investments, while Berk and Green (2004) propose a cross-sectional approach.
remain partially under-diversified and thus impose under-diversification to non-indexers as a group. Since in equilibrium market must clear, if a group of investors chooses to diverge from optimal diversification, it imposes under-diversification to remaining investors who as a group must hold the remaining available shares.

There is a growing literature presenting empirical evidence of effects related to indexing (see Wurgler (2011) for a summary). Besides comovement, the most widely studied effect is the presence of abnormal returns surrounding index additions and deletions. Shleifer (1986) provides evidence that these price increases do not fully revert back to the initial level. My model is consistent with this empirical observation. The comparative statics analysis shows that Sharpe ratios are larger for non-index stocks than for index stocks and that this effect is amplified as the relative wealth of indexers increases, therefore index stock prices will be higher.

3 An economy with indexers

The model consists of an infinite-horizon, continuous-time economy with three Lucas trees and two agent types, active investors and indexers. I extend Bhamra (2007), in which there are two agents and two risky assets, by adding an indexing constraint and an additional risky stock. This additional risky asset allows me to derive closed-form approximations for the comovement of an index stock with other index stocks and with non-index stocks as a function of state variables. One of these state variables is the relative wealth of indexers in the economy.

The indexing constraint is exogenous. One might think of indexing as a self-imposed constraint motivated by the simplicity of index-linked investing. Veldkamp (2006) presents an information-based explanation; she shows that in a Grossman-Stiglitz framework with economies of scale in information production, it is rational for some agents to only acquire information on an index rather than on individual stocks. These agents then endogenously choose to engage in index-linked investing, as they cannot distinguish between individual stocks.

3.1 Information structure

Uncertainty is represented by a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) on which is defined a 3-dimensional vector of independent Brownian motions \(Z = [Z_1 \ Z_2 \ Z_3]'\). The filtration \(\mathbb{F} = \{\mathcal{F}_t\}\) is the augmentation under \(\mathbb{P}\) of the filtration generated by \(Z\). The sigma-field \(\mathcal{F}_t\) represents the information available at time \(t\) and the probability measure \(\mathbb{P}\) represents the agent’s common beliefs. Stochastic processes to follow are progressively measurable with respect to \(\mathbb{F}\) and equalities involving random variables hold \(\mathbb{P}\)-a.s.

Peng and Xiong (2006) also find that categorizing assets is an endogenous outcome when investors have limited attention, as they choose to focus on market-wide and sector-wide information. Mondria (2010) presents another model where frictions on information lead to categorizing and generate price comovement.
3.2 Consumption space

There is a single perishable good, the numeraire. The agents’ consumption set \( C \) is given by
the set of non-negative progressively measurable consumption rate process \( c_t \) with \( \int_{0}^{T} |c_t| dt < \infty, \forall T \in [0, \infty) \).

3.3 Securities market

The investment opportunities are represented by a locally riskless bond earning the instantaneous interest rate \( r \) and three risky stocks, representing claims to exogenously given strictly positive dividend processes \( D_i, i = 1, 2, 3 \), with

\[
\frac{dD_{i,t}}{D_{i,t}} = \mu_{D}dt + \sigma_{D}dZ_{D,i,t}, \ i \in \{1, 2, 3\},
\]

where \( Z_{D,i} \) are standard Brownian motions\(^7\) with equal pairwise correlation coefficients \( \rho_{D} \).

The aggregate dividend is defined as \( D_{M,t} = \sum_{i=1}^{3} D_{i,t} \) and the index dividend is defined as \( D_{I,t} = \sum_{i=1}^{2} D_{i,t} \).\(^8\) The initial bond value is normalized to unity so that the bond price process is given by

\[
B_t = \exp \left( \int_{0}^{t} r_s ds \right).
\]

The stock price processes can be defined as

\[
dS_{i,t} = (S_{i,t} \mu_{i,t} - D_{i,t}) dt + S_{i,t} \sigma_{i,t} dZ_t,
\]

where \( \mu_t \) is the 3-dimensional column vector with \( \mu_{i,t} \) as the \( i \)th element and \( \sigma_t \) is the \( 3 \times 3 \) matrix with \( \sigma_{i,t} \) as the \( i \)th column. The instantaneous covariance matrix is \( \Sigma_t = \sigma_t \sigma_t' \). Both \( \mu_t \) and \( \sigma_t \) are determined endogenously in equilibrium.

Stock return processes can be defined from stock prices and dividends as

\[
dR_{i,t} = \frac{dS_{i,t} + D_{i,t} dt}{S_{i,t}} = \mu_{i,t} dt + \sigma_{i,t} dZ_t.
\]

The supply of each stock is normalized to one share, while the bond is in zero net supply.

There exist a value-weighted index with stocks 1 and 2 as its constituents. The third stock is a non-index stock.

\(^7\)\( Z_{D,i} \) are linear combinations of the fundamental independent Brownian motions \( Z_i \) defined in 3.1. For more details about the transformation, see Appendix A.11.\(^8\)See Appendix A.1 for details on the market and the index portfolios.
3.4 Trading strategies

Trading takes place continuously. An admissible trading strategy is a 4-dimensional vector process \((\alpha, \gamma)\), where \(\gamma\) is an \(n\)-dimensional column vector with \(\gamma_i\) as its \(i\)th element and \(\alpha_t\) and \(\gamma_{i,t}\) denote the amounts invested at time \(t\) in the bond and in stock \(i\), satisfying the required regularity conditions. A trading strategy \((\alpha, \gamma)\) is said to finance the consumption plan \(c \in C\) if the corresponding wealth process \(W = \alpha + 1'\gamma\) satisfies the dynamic budget constraint

\[
dW_t = [\alpha_tr_t + \gamma'_t \mu_t - c_t]dt + [\gamma'_t \sigma_t]dZ_t,
\]

where \(1\) is a 3-dimensional column vector of ones.

3.5 Agent’s preferences and endowments

There are two representative agents, an active investor \(A\) and an indexer \(I\), both with time-additive log-normal utility functions:

\[
U_{j,t}(c) = E_t \left[ \int_0^\infty e^{-\delta \tau} \log(c_{j,(t+\tau)}) d\tau \right], \ j \in \{A, I\}
\]

for some common rate of time preference \(\delta > 0\) and individual consumption \(c_j\).

Agents differ by their endowment and by their investment opportunity set. The indexer is endowed with a fraction \(\beta\) of each index stocks while the active investor owns \((1 - \beta)\) share of each index stock and one share of the non-index stock. The indexer faces an exogenous constraint that limits her investment opportunity set to the bond and the index portfolio, according to index weights, which are endogenous. The active investor is unconstrained and faces a complete market.

3.6 Equilibrium

Let \(E = ((\Omega, \mathcal{F}, \mathcal{F}, \mathcal{P}), D_1, D_2, D_3, U_1, U_2, \beta)\) denote the primitives for the economy. An equilibrium for the economy \(E\) is an interest rate stock price process \((r, S)\) and a set \(\{c_j^*, (\alpha_j^*, \gamma_j^*)\}\), \(j \in \{A, I\}\) of consumption and admissible trading strategies for the two agents such that

(i) \((\alpha_j^*, \gamma_j^*)\) finances \(c_j^*\) for \(j \in \{A, I\}\)

(ii) \(c_A^*\) maximizes \(U_A\) over the set of consumption plans \(c \in C\) which are financed by an admissible trading strategy \((\alpha, \gamma) \in \gamma\) with \(\alpha_0 + \gamma_0'1 = (1 - \beta)[S_{1,0} + S_{2,0}] + S_{3,0}\);

\[9\text{See pp.234-235 of Back (2010) for a formal presentation of the required regularity conditions.}\]
(iii) \( c^*_I \) maximizes \( U_I \) over the set on consumption plans \( c \in C \) which are financed by an admissible trading strategy \( (\alpha, \gamma) \in \gamma \) with \( \alpha_0 + \gamma_0' = \beta[S_{1,0} + S_{2,0}] \), \( \gamma_{j,t} = \gamma_{I,t} \frac{S_{j,t}}{S_{I,t} + S_{2,t}} \) for \( i = 1, 2 \) where \( \gamma_I \) is the amount invested in the index and \( \gamma_3 \equiv 0 \); and

(iv) all markets clear, that is, \( c^*_A + c^*_I = D, \alpha^*_A + \alpha^*_I = 0 \) and \( \gamma^*_A + \gamma^*_I = S \).

3.7 Relation to international asset pricing models

The setup is similar to models of international asset pricing with mild segmentation. It is conceptually related to a setup with two countries and with investors in one country being constrained to invest at home while the investors in the other country are unconstrained, Bhamra (2007), which I extend, is one such model. In my model, the constrained investor faces the additional constraint of having to invest according to the index weights; i.e. her actions influence the price of the index but not the price of index stocks relative to one another. My model is also closely related to Pavlova and Rigobon (2008), in which there are three countries (with three agents and three goods). They find that portfolio restrictions and terms of trade generate excess comovement in asset prices. While their portfolio constraints are different, the economic mechanism through which excess comovement appears is the same as in my model.

3.8 Agents’ Problem

Agent \( j \) has initial wealth \( W_{j,0} = \pi'_{j,0} S_0 \), where \( \pi_j \) is a 3-dimensional vector with the \( i \)-th element equal to agent’s \( j \) proportion of wealth invested in stock \( i \) at time \( t \) (\( \pi_j = \frac{\gamma_I}{W_{I,t}} \)). For the constrained indexer \( I \), this reduces to \( \pi_{I,t} = \pi_{I,t}\frac{[\omega_{1,t}^I \omega_{2,t}^I 0]}{\pi_{I,t}^{I',t} \pi_{I,t}^{I,t}} \), where \( \pi_{I,t}^{I',t} \) is the indexer’s proportion of wealth invested in the index portfolio. It will be more convenient to deal with normalized wealth such that \( \nu_{j,t} = \frac{W_{j,t}}{W_{J,t} + W_{I,t}} \) is the share of wealth owned by agent \( j \).

Agent \( j \)’s optimization problem at time \( t \) is to maximizes her time additive utility

\[
U_{j,t} = E_t \left[ \int_t^\infty e^{-\delta(s-t)} \log(c_{j,s})ds \right]
\]

subject to her budget constraint, which gives

\[
\max U_{j,t} \text{ subject to } E_t \left[ \int_0^\infty \frac{\xi_{j,s}}{\xi_{j,t}} c_{j,s} ds \right] \leq W_{j,t},
\]

where \( \xi_{j,t} \) is the marginal utility of consumption of agent \( j \) at time \( t \).

This process can be written as

\[
\frac{d \xi_{j,t}}{\xi_{j,t}} = -r_{j,t} dt - \theta'_{j,t} dZ_t,
\]
where $r_{j,t}$ is agent’s $j$ implied risk-free rate and $\theta_{j,t}$ is her shadow price of risk.

Both agents trade in the bond, so in equilibrium they will have the same risk-free rate (i.e. $r_{I,t} = r_{A,t} = r_t$). However their different investment opportunity sets means they will face different market prices of risk.

Agent $A$ is unconstrained, so her optimal portfolio proportions are given by

$$\pi_{A,t} = \Sigma_t^{-1}(\mu_t - r1).$$

(10)

For agent $I$, using the convex duality approach of Cvitanić and Karatzas (1992) to derive optimal portfolio weights,\(^{10}\) $\pi_{I,t}$ coincides with the optimal portfolio in the incomplete market:

$$\pi_I = \begin{bmatrix} \pi^{I}_{1} \omega^{I}_{1} \\ \pi^{I}_{2} \omega^{I}_{2} \\ 0 \end{bmatrix}$$

(11)

where $\pi^{I}_{I,t} = (\mu_{I,t} - r) / \sigma^2_{I,t}$.

The market clearing condition imposes that

$$\omega_t = \pi_{A,t}\nu_{A,t} + \pi_{I,t}\nu_{I,t},$$

(12)

where $\omega_t$ is a 3-dimensional vector with the $i$-th element equal to the value-weight of stock $i$ in the economy ($\omega_i = S_i / \sum_{k=1}^{3} S_k$). Substituting the optimal portfolio weights in the market clearing condition yields the following proposition:

**Proposition 1** In equilibrium, expected excess stock returns are as follows:

$$\mu_1 - r_f = \frac{1}{\sigma^2_{I,t}}[(\mu_I - r_f)(\sigma_1\omega_1 + \rho_{1,2}\sigma_2\omega_2)$$

$$+ \frac{\omega_2\omega_3}{\nu_{A}\omega_I}\left(\omega_I[\text{cov}(R_1, R_2)\text{cov}(R_1, R_3) - \sigma^2_2\text{cov}(R_2, R_3)]ight)$$

$$- \omega_2[\text{cov}(R_1, R_2)\text{cov}(R_2, R_3) - \sigma^2_2\text{cov}(R_1, R_3))\]]$$

$$\mu_3 - r_f = \omega_3\sigma^2_3 + (1 - \omega_3)\text{cov}(R_I, R_3) + \omega_3\sigma^2_3\left[\frac{\nu_I}{\nu_{A}}(1 - \rho_{I,3}^2)\right]$$

$$\mu_I - r_f = \omega_I\sigma^2_I + (1 - \omega_I)\text{cov}(R_I, R_3)$$

(13)

(14)

(15)

where $\mu_I$ and $\sigma_I$ denote the drift and variance of the index and $\omega_I = \omega_1 + \omega_2$.

Result for stock 2 is omitted as it is symmetric to stock 1.

\(^{10}\)See Appendix A.2 for details. This methodology has been previously used in the finance literature for portfolio constraints, see for example Shapiro (2002) and Pavlova and Rigobon (2008) for similar applications or Chapters 5 and 6 of Karatzas and Shreve (1998) for a textbook treatment.
Proof See Appendix A.4

Proposition 1 tells us that holding variances/covariances constant, the non-index stock excess returns are increasing in the relative wealth of passive investors. This is due to the additional risk active investors must take as they become more under-diversified (as compared with the case where they hold the market). It is highlighted by the term of correlation between the index and stock 3. This is consistent with the standard result from one period models of mild segmentation. I cannot however conclude from Proposition 1 on the actual equilibrium effect of an increase in $\nu_I$ as the variance and covariance terms are determined endogenously in equilibrium and thus also depend on $\nu_I$.

3.9 Representative agent

The equilibrium concept used is an extension of Lucas [1978] in which I determine the equilibrium asset prices as the shadow prices from the pricing kernel determined by the investor’s marginal utility of consumption and market clearing. Market clearing implies that they must hold all the shares in the world, with the restriction that the indexer is constrained in her investment opportunity set, therefore forcing the unconstrained agent to hold all the shares of the non-index stock.

Following Cuoco and He [1994], I can still use a social planner to derive equilibrium prices, but the weight $\lambda_t$ will be stochastic:

$$U_t = E_t \int_t^{\infty} e^{-\delta(s-t)} (\log c_{A,s} + \lambda_s \log c_{I,s}) ds. \quad (16)$$

Agent $A$ is unconstrained, so her state-price density must correspond to the equilibrium state-price density:

$$\xi_t = \xi_{A,t} = \kappa_A e^{-\delta t (\nu_{A,t} D_{M,t})^{-1}} \quad (17)$$

Solving for the equilibrium state-price density in terms of endogenous stock returns moments, I obtain the following proposition:

Proposition 2 The equilibrium risk-free rate and price of risk of the representative agent are

$$r_f = r_{f}^* + \frac{\rho_{\nu_A D_M} \sigma_{\nu_A} \sigma_{D_M}}{\nu_A}, \quad (18)$$

$$\theta = \theta^* + \frac{\sigma_{\nu_A}}{\nu_A}. \quad (19)$$

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11 See, e.g., Errunza and Losq [1985].
13 See Appendix A.2 for details of the derivation.
where \( r_f^* = \delta + \mu_D - \sigma_D^2 \) and \( \bar{\theta}^* = \bar{\sigma}_D \) are the risk-free rate and price of risk in the unconstrained economy (\( \bar{\theta} \) and \( \bar{\sigma} \) indicate 3-dimensional vectors) and \( \rho_{\nu A D_M} \) is the covariance between the aggregate dividend process and the consumption share process \( \nu_A \).

**Proof** See Appendix A.5 for the derivation. From Corollary 2 it will be clear that when \( \nu_A = 1 \) (there is no indexer) the term \( \sigma_{\nu A} \) vanishes and we obtain \( r_f^* \) and \( \bar{\theta}^* \).

Proposition 2 states that the risk-free rate is increasing in \( \nu_I \) if the covariance between the relative wealth process of the active investor with the aggregate dividend is positive. While I do not prove it formally, numerical results show that, with reasonable parameters, \( \rho_{\nu A D_M} \) is positive; the risk-free rate is increasing in \( \nu_I \) as the indexer is a net borrower. The proposition also states that the price of each risk factor is increasing in the sensitivity of the relative wealth process to each risk factor and in \( \nu_I \). However, since the relative wealth process is endogenous to the model, it is not yet clear what the effect will be in equilibrium. The following corollary will shed a little more light on these results.

**Corollary 1** In equilibrium, the prices of risk of individual agents have the following relationship:

\[
\left( \bar{\theta}_A - \bar{\theta}^* \right) = -\frac{\nu_T}{\nu_A} \left( \bar{\theta}_I - \bar{\theta}^* \right)
\]  

(20)

**Proof** See Appendix A.6.

Corollary 1 illustrates where the excess comovement originates. Keeping in mind that \( \nu_I \) and \( \nu_A \geq 0 \), it states that individual agents prices of risk will deviate from the unconstrained price of risk in opposite directions. In equilibrium, the components of \( \bar{\theta}_I \) corresponding to the index stocks are larger than the corresponding components of \( \bar{\theta}^* \), while the opposite is true for the component corresponding to the non-index stock. Any shock to the relative wealth of the two agents affects the equilibrium price of risk \( \bar{\theta}_A \) in opposite directions for index and non-index stocks. This generates excess comovement. The slope of the coefficient \( \frac{\nu_T}{\nu_A} \) is \( (1 - \nu_T)^{-2} \), so it is increasing in \( \nu_T \). This causes the excess comovement to increase with relative wealth of indexers.

**Corollary 2** In equilibrium, the share of aggregate wealth owned by the active investor follows the process

\[
d\nu_A = \mu_{\nu A} dt + \sigma_{\nu A} dZ_\lambda
\]  

(21)

where

\[
\mu_{\nu A} = \nu_A \nu_I^2 \sigma_\lambda^2,
\]  

(22)

\[
\sigma_{\nu A} = \nu_A \nu_I \sigma_\lambda,
\]  

(23)
and $\sigma_\lambda$ is the volatility of the stochastic weight in the representative agent’s problem.

**Proof** Follows from the proof of Proposition 2.

Corollary 2 illustrates that the equilibrium is not stationary. Since $\mu_{\nu_A}$ is positive, over time the active investor will dominate, and $\nu_A = 1$ ($\nu_A = 0$) is an absorbing state (in that case both $\mu_{\nu_A}$ and $\sigma_{\nu_A}$ are equal to 0). This is expected from a model with constrained investors; the unconstrained one will dominate over time since both agents have the same preferences but differ in their investment opportunity sets. Thus, in its current form the model cannot explain the rise of indexing of the past decades. A richer model could generate the observed level of indexing as an endogenous outcome in a general equilibrium setup. For example, this could be done by adding frictions such as the incremental cost of active investing and allowing one rational agent to invest both passively and actively at same time, as in Petajisto (2009). However, the additional complexity is not necessary for the current discussion. The current form of the model provides valuable insight on stock prices dynamics for given levels of indexing, which is the purpose of this paper.

### 3.10 Stock return dynamics

Since both agents have time-additive log utility, it follows from the first order condition of the HJB equation that the aggregate stock market value $S_{M,t} = S_{1,t} + S_{2,t} + S_{3,t} = D_{M,t}/\delta$, and thus aggregate stock market value is independent of the relative wealth of agents.

**Proposition 3** The equilibrium stock prices have the form

$$S_{i,t} = S^A_{i,t} + \frac{\nu_{\lambda,t}}{\nu_A,t} X^T_{i,t},$$

where $S^A_{i,t}$ is the stock price in the unconstrained economy and $X^T_{i,t}$ is an adjustment for the consumption of the constrained agent.

**Proof** See Appendix A.7.

Using a perturbation expansion I obtain a closed form approximation for the stock price in the unconstrained economy:

$$S^A_1 = \frac{D_M}{\delta} \left( s_1 + s_1 (1 - 3s_1 + 2s_1^2 - 2s_2 + 2s_1s_2 + 2s_2^2) \left(1 - \rho_D\right) \sigma_D^2 \right) + O(\epsilon^4),$$

---

14 For infinite horizon log utility, $J(t, w, x) = \log w + f(t, x)$. The FOC of the HJB equation is $u'(c_i) = J_{w_i}$, which yields $c_i = w_i \delta$. In the current setup, the representative agent must consume the aggregate dividend and own the aggregate stock market, thus $c_A + c_T = D_M$ and $w_A + w_T = S_M$. We thus have that $S_M = D_M/\delta$.

15 See Hinch (1991) for a textbook treatment of perturbation methods.
where \( s_i = D_i / D_M \) is the share of dividends of asset \( i \). I find \( S_A^2 \) by symmetry and \( S_A^3 \) by using the fact that \( S_A^3 = D_M \delta - S_A^1 - S_A^2 \).

Solving for \( X^T_{i,t} \) is trickier as the relative wealth process \( \nu_T \) depends on stock prices and stock prices depend on \( \nu_T \), so I must solve for \( \nu_T \) and \( X^T_{i,t} \) jointly. Using again a perturbation expansion, I obtain a closed form approximation for the stock price adjustment in the constrained economy:

\[
X^T_1 = \frac{D_M s_1}{2(s_1 + s_2) \delta^2} \left( 2(s_1 + s_2 - 1) \left( 2 \left( s_1^2 - s_1 + s_1 s_2 + s_2^2 \right) - s_2 \right) + (s_1 + s_2) \left( 1 + 2s_1^2 - 3s_1 + 2s_1 s_2 + 2s_2^2 - 2s_2 \right) \nu_A \right) (1 - \rho_D) \sigma_D^2 + O(\epsilon^4). \tag{26}
\]

As for the stock prices in the unconstrained economy, I find \( X^T_2 \) by symmetry and \( X^T_3 \) by \( X^T_1 + X^T_2 + X^T_3 = 0 \).

### 4 Indexing and stock return dynamics

I present predictions of the model regarding dynamics of stock returns in the presence of indexers, which I obtain from numerical results. Stock return dynamics are function of the relative level of each dividend stream and of the relative wealth of the two agents. Since I am mostly concerned with the effect of the relative weight of indexers in the economy, I fix the relative level of each dividend stream and look at conditional moments as a function of the relative weight of indexers \( \nu_T \).

#### 4.1 Parameter values

For the following discussion, I fix the parameter values. I set the rate of time preference \( \delta = 0.01 \). I obtain from Datastream the historical dividend yield for the S&P 500 and the S&P 600 Smallcap from 1999 to 2011. The correlation between the two series is 0.956 using annual data and 0.919 using daily data, so I set \( \rho_D = 0.94 \). The average dividend yield for the S&P 500 from 1983 to 2011 is 2.60%, so I set \( \mu_D = 0.026 \). The standard deviation of annual returns for the S&P 500 from 1983 to 2011 is 0.17, so I set \( \sigma_D = 0.17 \). Cochrane, Longstaff, and Santa-Clara (2008) show that in a model with fixed supply of assets, the dividend share influences stock return dynamics, so I look at three configurations for the relative level of dividend streams. In each of them, I set the current level of stock 1’s dividend to 1% of the current aggregate dividend (\( s_1 = 0.01 \)), so it represents a marginal index stock. In the first configuration, I set the level of stock 2 and 3’s dividends to equal shares of the rest (\( s_2 = s_3 = 0.495 \)). In an unconstrained economy, stocks 2 and 3 would have the same dynamics since their dividend processes have the same parameters and their dividend levels are equal. However, this assumption of equal shares is far from the reality of the two most widely used US equity indices: the S&P 500 and the
Russell 2000. The relative weight of the index in the economy is an important factor to consider when looking at implications of indexing. As of 2010, the S&P 500 represents about 78% of the total US market capitalization, while the Russell 2000 represents only 9%. The second and third specifications are set to mimic the S&P 500 and Russell 2000, with respective dividend levels set to \( s_2 = 0.77, s_3 = 0.22 \) and \( s_2 = 0.08, s_3 = 0.91 \). Stock prices also depend on the current level of aggregate dividends, so I normalize \( D_M = 1 \) in order to get stock prices that reflect a relative weight of each stock’s value in the market \( (S_M = D_M/\delta = 100) \). The focus of this analysis should be qualitative rather than structural. I do not claim that the model quantitatively matches empirical evidence. As illustrated by Petajisto (2009), it is not possible to generate indexing effects of economically significant magnitude with this type of model.

4.2 Stock prices

Figure 3 shows the stock price for the large index stock and non-index stock as a function of \( \nu_I \) for the equal size specification. As the effect of segmentation becomes larger, the index stock increases in price while the non-index stock decreases in price. This illustrates that stocks are imperfect substitutes as one is not available to all investors. This is consistent with Shleifer (1986) who finds that stocks included in the S&P 500 experience positive abnormal returns on the event and that there is a permanent component to that effect. The results are qualitatively similar for the other specifications, but the price differential due to the dividend share differential makes it hard to see on a graph.

Figure 3

Stock price of the large index stock (stock 2, solid blue line) and the non-index stock (stock 3, dashed red line), as a function of \( \nu_I \), the share of wealth owned by the indexer. The parameters used are \( \delta = 0.01 \), \( \sigma_D = 0.17 \), \( \mu_D = 0.026 \), \( \rho_D = 0.94 \) and \( D_M = 1 \). Results are for the specification where the large index stock (stock 2) and the large non-index stock (stock 3) have the same dividend level \( (s_2 = s_3 = 0.495) \).

Figure 4 shows that the expected return of the index stock is decreasing as a function of \( \nu_I \)
for all specifications while it is increasing for the non-index stock, which explains the relationship
for prices. This decrease in expected return leads to a decrease in the Sharpe ratio for the index
stock. Depending on the relative dividend shares, the Sharpe ratio of the non-index stock is
increasing or decreasing with $\nu_I$, but the slope is always negative and of greater magnitude for
the index stock.

### 4.3 Comovement and the price of risk

The idea of excess comovement is that index stocks comove more with other index stocks than
they do with non-index stocks compared to the case with no indexers. Figure 5 shows the
comovement $\beta$ of index stock 1 with the other index stock and with the non-index stock, as
measured with a bivariate regression in the style of Barberis, Shleifer, and Wurgler (2005). We can see that for each specification, the $\beta$ with the other index-stock increases with $\nu_I$ while the $\beta$ with the non-index stock decreases. The origin of the x axis ($\nu_I = 0$) shows the comovement in an unconstrained model. Deviations from those values as $\nu_I$ increases represent excess comovement. The model predicts that excess comovement is increasing in the relative wealth of the indexer.

To understand what is driving this excess comovement, it is useful to look at the effect of
indexers on the price of risk $\theta$. The components of $\theta$ represent the price of risk factors $Z_i$, which
are the shocks to the dividend streams. $Z_1$ and $Z_2$ are the shocks to the index stocks’ dividend
streams while $Z_3$ is the shock to the non-index stock’s dividend stream. Figure 6 presents the
price of risk for a setup with equal dividend share between stock 2 and 3. Panel (a) presents
the price of risk in the unconstrained economy (does not depend on $\nu_I$) and Panel (b) presents
the shadow price of risk for the indexer. As we can see, the indexer has a much higher shadow
price of risk for the index stocks than in the unconstrained case, while the opposite is true
for the non-index stock. The resulting equilibrium price of risk, presented in Panel (c), is one
where the price of risk of $Z_1$ and $Z_2$ are decreasing as $\nu_I$ increases, while the price of risk of
$Z_3$ increases. This means that for a given shock to the relative wealth of the indexer, the price
of risks associated with index stocks reacts in the opposite direction than the one associated
with the non-index stock, generating excess comovement. Furthermore, the magnitudes of the
slopes are increasing in $\nu_I$, so the resulting excess comovement becomes larger as $\nu_I$ increases.

---

16 The values used are $s_1 = 0.1$ and $s_2 = s_3 = 0.45$. $s_1$ is set to a larger value than in the previous figures to scale
the associated price of risk on the graph, but the otherwise the results would be similar with $s = 0.01$.

17 From [20], we know that the slope is actually $-(\bar{\theta}_I - \bar{\theta})/(1 - \nu_I)$. 

---

17
Figure 4
Expected returns and Sharpe ratios of the large index stock (stock 2, solid blue line) and the non-index stock (stock 3, dashed red line), as a function of $\nu_I$, the share of wealth owned by the indexer. As the relative wealth of the indexer becomes larger, the Sharpe ratio of the index stock decreases. The parameters used are $\delta = 0.01$, $\sigma_D = 0.17$, $\mu_D = 0.26$ and $\rho_D = 0.94$. Panels (a) and (d) presents results for the setup where the large index stock (stock 2) and the large non-index stock (stock 3) have the same dividend level ($s_2 = s_3 = 0.495$). Panel (b) and (e) present results for the setup where the large index stock has a higher dividend level than the large non-index stock ($s_2 = 0.77$, $s_3 = 0.22$). Panel (c) and (f) present results for the setup where the large index stock has a lower dividend level than the large non-index stock ($s_2 = 0.08$, $s_3 = 0.91$).
Figure 5
Comovement of the marginal index stock (stock 1) with the large index stock (solid blue line) and the non-index stock (dashed red line), as a function of $\nu_I$, the share of wealth owned by the indexer. As the relative wealth of the indexer becomes larger, the comovement of the index stock with the other index stock increases while the comovement of the index stock with the non-index stock decreases. The parameters used are $\delta = 0.01$, $\sigma_D = 0.17$, $\mu_D = 0.26$ and $\rho_D = 0.94$. Panel (a) presents results for the setup where the large index stock (stock 2) and the large non-index stock (stock 3) have the same dividend level ($s_2 = s_3 = 0.495$). Panel (b) presents results for the setup where the large index stock has a higher dividend level than the large non-index stock ($s_2 = 0.77$, $s_3 = 0.22$). Panel (c) presents results for the setup where the large index stock has a lower dividend level than the large non-index stock ($s_2 = 0.08$, $s_3 = 0.91$).

Figure 6
Price of risk for each risk factor, as a function of $\nu_I$, the share of wealth owned by the indexer. The green and blue dashed lines correspond to the price of risks associated $Z_1$ and $Z_2$ respectively (shocks to index stocks dividends) while the solid red line corresponds to the price of risk associated with $Z_3$ (shocks to the non-index stock dividends). As the relative wealth of the indexer becomes larger, the comovement of the index stock with the other index stock increases while the comovement of the index stock with the non-index stock decreases. The parameters used are $\delta = 0.01$, $\sigma_D = 0.17$, $\mu_D = 0.26$, $\rho_D = 0.94$ $s_1 = .1$ and $s_2 = s_3 = .45$. Panel (a) presents $\theta^*$, which corresponds to the price of risk in the unconstrained economy. Panel (b) presents the shadow price of risk for the indexer ($\theta_I$). Panel (c) presents the equilibrium price of risk.
5 Indexing and comovement in the S&P 500

Using S&P 500 stocks, I test empirically the prediction that index stocks comove more with other index stocks and less with non-index stocks as the relative wealth of indexers increases.

5.1 Data and variable construction

I obtain S&P 500 index constituents and individual stock returns and characteristics from CRSP. I consider all CRSP stocks listed on Amex, Nasdaq or NYSE with share codes 10 or 11. These restrictions exclude an average of 16 S&P 500 stocks per year after 1983. The annual estimates of passive assets under management in the S&P 500 for 1983-2010 are from Standard and Poor’s. I define the passive ownership (PO) of an index at year $t$ as the total passive assets under management divided by the total CRSP market capitalization. I use this value as a proxy for the relative wealth of passive investors. Figure 7 presents the passive ownership of the S&P 500 for 1983 to 2010.

![Passive Ownership Graph](image-url)

**Figure 7**
End of year passive ownership for the S&P 500. Passive ownership is defined as passive assets under management for the S&P 500 divided by total CRSP market capitalization. Passive assets under management is from Standard and Poor’s and market capitalization is from CRSP.
5.1.1 Average comovement of index stocks

I first estimate comovement from 1957, the year the S&P 500 was first published, and then restrict the study on the relationship with passive ownership to 1983-2011 (the years for which the data is available). Following Barberis, Shleifer, and Wurgler (2005), I estimate comovement of an index stock by regressing its returns on the returns of the index (net of the effect of the stock) and the market returns net of the index:

\[
R_{j,t} = \alpha_j + \beta_{j,SP500} R_{SP500,t} + \beta_{j,nonSP500} R_{nonSP500,t} + u_{j,t}. \tag{27}
\]

I compute value-weighted daily returns of the S&P 500 portfolio (excluding the stock under study) and of the non-S&P 500 portfolio using data from CRSP. This differs from the way Barberis, Shleifer, and Wurgler (2005) compute returns: they back out individual effect of the stock from S&P 500 returns. My approach avoids the need to estimate the daily weight of each index stock in order to remove its net contribution. It also keeps the weighting methodology constant throughout the sample, while the S&P 500 changed from value-weighted to float-weighted in 2005.

For every year, I look at stocks that are in the index at the end of December. I estimate comovement \( \beta \)'s using daily data for the months \([+1, +12]\), with a minimum of 6 months availability for the estimate to be considered valid. Figure 1 (in the introduction) presents the time series of average comovement \( \beta \)'s for the S&P 500. Consistent with predictions of the model, we can see that for almost all months since indexing has become non-negligible (early 1980s), the average comovement of index stocks with the index is greater than with the rest of the market. This is also consistent with Barberis, Shleifer, and Wurgler (2005) and Boyer (2011) who both find the a similar effect while focusing only on stocks that get added or removed from an index. Their focus is on the change in comovement surrounding index additions and deletion events, not on the time series of comovement.

The model predicts that the \( \beta \) with the index should be increasing in passive investors’ wealth. It also predicts the opposite relationship for the \( \beta \) with the rest of the market. I use passive ownership (PO) as a proxy for passive investors’ wealth.

5.2 Passive ownership and comovement

The goal of Barberis, Shleifer, and Wurgler (2005) is to investigate whether the excess comovement of index stocks with other index stocks is caused by the index membership, so in their setup an event-study is appropriate. They find that after inclusion index stocks comove more with other index stocks and less with non-index stocks. These results are stronger later in their sample, supporting the idea that comovement is caused by indexing. My objective is to
determine if this effect becomes stronger as the weight of indexers increases.

I test this hypothesis in a more direct manner, using regressions based on the average comovement of S&P 500 stocks. Looking at Figure 7, one may have reasonable concerns that the increase in comovement is due to a time trend that just happens to be correlated with the increase in passive ownership. In order to avoid this issue, the analysis focuses on absolute changes in comovement and in incremental passive ownership. I regress changes in average $\beta_{SP500,t}$ and average $\beta_{nonSP500,t}$ (which I denote $\Delta\beta_{SP500,t}$ and $\Delta\beta_{nonSP500,t}$) as dependent variables and the lagged changes in the index’s passive ownership ($\Delta PO_{t−1}$) as the independent variable. Changes in $\beta$ exhibit negative auto-correlation, so I also repeat each regression including the lagged value of the dependent variable ($\Delta\beta_{t−1}$, which corresponds to $\Delta\beta_{SP500,t−1}$ or $\Delta\beta_{nonSP500,t−1}$.) Changes are computed as the absolute difference between two annual observations.

Table 1 presents regressions results. The null hypothesis is that all the coefficients are 0. The model predicts a positive sign for the coefficient on $\Delta PO_{t−1}$ in the regression of $\Delta\beta_{SP500}$ and a negative sign for the coefficient on $\Delta PO_{t−1}$ in the regression of $\Delta\beta_{nonSP500}$. Coefficients on $\Delta PO_{t−1}$ are all of the expected signs and statistically significant. Furthermore, the adjusted $R^2$ are quite high, suggesting that the change in passive ownership explain a good fraction of variation. This suggests that S&P 500 stocks do comove more with other S&P 500 stocks and less with non-S&P 500 stocks as indexer wealth increases.

5.3 Robustness checks

5.3.1 Panel regression

The previous regression results are strong, but one might wonder if the relationship is also present in the cross-section of comovement $\beta$s. In repeat the previous tests using a pooled regression with individual firms’ changes in comovement $\beta$ as dependent variables. I also include firm-specific lagged changes in $\beta$ as control variable. I include firm-fixed effects and cluster standard errors by year. Table 1 presents regressions results. Considering all observation within a year share the same change in passive ownership, it would be unlikely for $\Delta PO_{t−1}$ to explain much of the variation. This is evident from the low $R^2$. Nonetheless, the coefficients $\Delta PO_{t−1}$ are of the same sign and magnitude as in the regression based on mean $\beta$s. They are also statistically significant, with the exception of the last regression.

5.3.2 Size effects

Size has long been considered a driver of stock returns. If large firms are indeed fundamentally similar, we would expect them to comove more with other large firms even in the absence of
Results for regressions of changes in comovement estimates ($\Delta \beta_{SP500,t}$ and $\Delta \beta_{nonSP500,t}$) on the lagged changes in passive ownership ($\Delta PO$), from 1983 to 2010 (annual), and on $\Delta \beta_{t-1}$ which is the lagged value of the dependent variable in the regression, for S&P 500 firms. The dependent variable are the changes in average $\beta$s. Comovement $\beta$s are estimated from the regression

$$R_{j,t} = \alpha_j + \beta_{j,SP500,t}R_{SP500,t} + \beta_{j,nonSP500,t}R_{nonSP500,t} + u_{j,t},$$

where $R_{SP500,t}$ is the value-weighted return of the S&P 500 stocks portfolio (excluding stock $j$) and $R_{nonSP500,t}$ is the value-weighted return of the rest of the market. Comovement $\beta$s are estimated based on index membership at the end of December, using daily data for the following 12 months. PO is the total passive assets under management divided by the total CRSP market capitalization and acts as a proxy for the relative wealth of indexers. T-stats are presented in parenthesis. I use heteroscedasticity-consistent standard errors. *, **, *** indicate significance at 10%, 5% and 1% level respectively.

<table>
<thead>
<tr>
<th>Dep. var.</th>
<th>$\Delta \beta_{SP500,t}$</th>
<th>$\Delta \beta_{nonSP500,t}$</th>
</tr>
</thead>
<tbody>
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<td>Intercept</td>
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</tr>
<tr>
<td></td>
<td>(-0.88)</td>
<td>(0.18)</td>
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<tr>
<td>$\Delta PO_{t-1}$</td>
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<td>8.05**</td>
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<td>(2.24)</td>
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<tr>
<td>$\Delta \beta_{t-1}$</td>
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<td>(-1.72)</td>
</tr>
<tr>
<td>Adj. $R^2$</td>
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<td>0.07</td>
</tr>
<tr>
<td>Obs.</td>
<td>26</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 2

Results for regressions of changes in comovement estimates ($\Delta \beta_{SP500,t}$ and $\Delta \beta_{nonSP500,t}$) on the lagged changes in passive ownership ($\Delta PO$), from 1983 to 2010 (annual), and on $\Delta \beta_{t-1}$ which is the lagged value of the dependent variable in the regression, for S&P 500 firms. The pooled regressions use individual firm changes in $\beta$s as the dependent variable. Comovement $\beta$s are estimated from the regression

$$R_{j,t} = \alpha_j + \beta_{j,SP500,t}R_{SP500,t} + \beta_{j,nonSP500,t}R_{nonSP500,t} + u_{j,t},$$

where $R_{SP500,t}$ is the value-weighted return of the S&P 500 stocks portfolio (excluding stock $j$) and $R_{nonSP500,t}$ is the value-weighted return of the rest of the market. Comovement $\beta$s are estimated based on index membership at the end of December, using daily data for the following 12 months. PO is the total passive assets under management divided by the total CRSP market capitalization and acts as a proxy for the relative wealth of indexers. T-stats are presented in parenthesis. Standard errors are clustered by year and include firm fixed effects. *, **, *** indicate significance at 10%, 5% and 1% level respectively.

<table>
<thead>
<tr>
<th>Dep. var.</th>
<th>$\Delta \beta_{SP500,t}$</th>
<th>$\Delta \beta_{nonSP500,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta PO_{t-1}$</td>
<td>7.14**</td>
<td>6.69*</td>
</tr>
<tr>
<td></td>
<td>(2.16)</td>
<td>(1.84)</td>
</tr>
<tr>
<td>$\Delta \beta_{t-1}$</td>
<td>-0.46***</td>
<td>-0.46***</td>
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<tr>
<td></td>
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<td>$R^2$</td>
<td>0.00 0.00 0.00 0.00 0.00 0.00</td>
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<tr>
<td>Obs.</td>
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</tr>
</tbody>
</table>
indexers. Since the S&P 500 is a value-weighted index,\footnote{The S&P 500 switched to float-weighting in 2005. Float weighting uses the market capitalization of the float – the shares available for trading – excluding shares held by large block-holders.} comovement with the index means comovement with large S&P 500 stocks. If large S&P stocks indeed have comoving fundamentals, we should expect the effect of indexers on those stocks to be weakened or even negligible. On the other hand, if small S&P 500 stocks – which are still large stocks by Fama-French standards – have fundamentals that comove more with stocks of similar size, including non-S&P 500 stocks like the large S&P 400 MidCap stocks, we should expect the effect of indexers to be more pronounced on them. The value-weighted $R_{\text{nonSP500}}$ put more weight on the biggest non-S&P 500 stocks, which are more likely to be close in size to small S&P 500 stocks.

To look at differences due to size, I split the sample of S&P 500 stocks in two. Each year, the top portfolio includes the largest 20 stocks, which account on average for 32.0% of the index market cap, and the bottom portfolio includes the remaining stocks. Average comovement $\beta$s for both subsamples are presented in Figure 8. Clearly, it does appear that index comovement was higher for larger stocks than for smaller stocks at the beginning of the sample, when indexing was marginal. I repeat the previous tests on stocks in the bottom portfolio, thus excluding the largest 20 stocks from the analysis and present the results in Table 3. The coefficients have the expected signs, are slightly more statistically significant and larger in magnitude than those for the full sample presented in Table 1 for the regressions on averages. Thus, results do not appear to be driven by the large S&P 500 stocks. Results for the top portfolio (not shown) are not statistically significant.

5.3.3 Correlated trading

In the model, comovement is a product of changes in discount rate. An alternative explanation for index-linked comovement is correlated trading. When news is made public, index stocks are traded more quickly to reflect the new information, while non-index stock may lag behind. Under this hypothesis, comovement should not be noticeable at lower frequencies. To test for this hypothesis, I repeat the tests using comovement based on weekly returns. The methodology for computing average comovement $\beta$s is the same as in Section 5.1.1 with the exception that I use weekly returns instead of daily returns. Figure 9 presents the time series of average comovement $\beta$s using weekly returns. The pattern is very similar to the one obtained with daily returns. Regression results are presented in Table 4. Even for comovement computed using weekly returns, the results are all of the expected sign and statistically significant. Note that this does not reject the correlated trading hypothesis. It merely illustrates that if correlated trading impacts comovement, the effect is likely orthogonal to the effect of passive ownership.
Average annual comovement $\beta$s from 1957 to 2010 estimated using a $[+1,+12]$ months window from the bivariate regression $R_{j,t} = \alpha_j + \beta_{j,SP500}R_{SP500,t} + \beta_{j,nonSP500}R_{nonSP500,t} + u_{j,t}$. Results are for stocks in the index on the last trading day of the year, using returns from the following year.
Results excluding the largest 20 firms in the S&P 500 for regressions of changes in comovement estimates ($\Delta \beta_{SP500,t}$ and $\Delta \beta_{nonSP500,t}$) on the lagged changes in passive ownership ($\Delta \PO$), from 1983 to 2010 (annual), and on $\Delta \beta_{t-1}$ which is the lagged value of the dependent variable in the regression, for S&P 500 firms. The dependent variable are the changes in average $\beta$s. Comovement $\beta$s are estimated from the regression $\hat{R}_{j,t} = \alpha_j + \beta_{j,SP500,t}R_{SP500,t} + \beta_{j,nonSP500,t}R_{nonSP500,t} + u_{j,t}$, where $R_{SP500,t}$ is the value-weighted return of the S&P 500 stocks portfolio (excluding stock $j$) and $R_{non-SP500,t}$ is the value-weighted return of the rest of the market. Comovement $\beta$s are estimated based on index membership at the end of December, using daily data for the following 12 months. $\PO$ is the total passive assets under management divided by the total CRSP market capitalization and acts as a proxy for the relative wealth of indexers. T-stats are presented in parenthesis. I use heteroscedasticity-consistent standard errors. *, **, *** indicate significance at 10%, 5% and 1% level respectively.

<table>
<thead>
<tr>
<th>Dep. var.</th>
<th>$\Delta \beta_{SP500,t}$</th>
<th>$\Delta \beta_{nonSP500,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-0.02</td>
<td>0.00 0.02</td>
</tr>
<tr>
<td></td>
<td>(-0.88)</td>
<td>(0.19) (-0.89) (0.70) (-0.10) (0.69)</td>
</tr>
<tr>
<td>$\Delta \PO_{t-1}$</td>
<td>7.86**</td>
<td>8.51** -11.11* -11.21*</td>
</tr>
<tr>
<td></td>
<td>(2.10)</td>
<td>(2.27) (-2.00) (-2.05)</td>
</tr>
<tr>
<td>$\Delta \beta_{t-1}$</td>
<td>-0.32</td>
<td>-0.36 -0.01 -0.04</td>
</tr>
<tr>
<td></td>
<td>(-1.17)</td>
<td>(-1.70) (-0.03) (-0.15)</td>
</tr>
<tr>
<td>Adj. $R^2$</td>
<td>0.12</td>
<td>0.07 0.22 0.12 -0.04 0.08</td>
</tr>
<tr>
<td>Obs.</td>
<td>26</td>
<td>26 26 26 26 26 26</td>
</tr>
</tbody>
</table>
Table 4

Results using weekly returns for regressions of changes in comovement estimates ($\Delta \beta_{SP500,t}$ and $\Delta \beta_{nonSP500,t}$) on the lagged changes in passive ownership ($\Delta PO$), from 1983 to 2010 (annual), and on $\Delta \beta_{t-1}$ which is the lagged value of the dependent variable in the regression, for S&P 500 firms. The dependent variable are the changes in average $\beta$s. Comovement $\beta$s are estimated from the regression $R_{j,t} = \alpha_j + \beta_{j,SP500,t}R_{SP500,t} + \beta_{j,nonSP500,t}R_{nonSP500,t} + u_{j,t}$, where $R_{SP500,t}$ is the value-weighted return of the S&P 500 stocks portfolio (excluding stock $j$) and $R_{non-SP500,t}$ is the value-weighted return of the rest of the market. Comovement $\beta$s are estimated based on index membership at the end of December, using daily data for the following 12 months. PO is the total passive assets under management divided by the total CRSP market capitalization and acts as a proxy for the relative wealth of indexers. T-stats are presented in parenthesis. I use heteroscedasticity-consistent standard errors. *, **, *** indicate significance at 10%, 5% and 1% level respectively.

<table>
<thead>
<tr>
<th>Dep. var.</th>
<th>$\Delta \beta_{SP500,t}$</th>
<th>$\Delta \beta_{nonSP500,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-0.02</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>(-0.67)</td>
<td>(0.31)</td>
</tr>
<tr>
<td>$\Delta PO_{t-1}$</td>
<td>12.36*</td>
<td>13.67**</td>
</tr>
<tr>
<td></td>
<td>(1.80)</td>
<td>(2.19)</td>
</tr>
<tr>
<td>$\Delta \beta_{t-1}$</td>
<td>-0.47**</td>
<td>-0.50***</td>
</tr>
<tr>
<td></td>
<td>(-2.49)</td>
<td>(-3.09)</td>
</tr>
<tr>
<td>Adj. $R^2$</td>
<td>0.07</td>
<td>0.20</td>
</tr>
<tr>
<td>Obs.</td>
<td>26</td>
<td>26</td>
</tr>
</tbody>
</table>
Estimates of comovement from weekly returns. Average annual comovement $\beta$s of S&P 500 stocks from 1957 to 2010 estimated using a $[+1, +12]$ months window from the bivariate regression $R_{j,t} = \alpha_j + \beta_{j,SP500}R_{SP500,t} + \beta_{j,nonSP500}R_{nonSP500,t} + u_{j,t}$. Results are for stocks in the index on the last trading day of the year, using returns from the following year.

5.3.4 Post-1990

Both comovement of index stocks and passive ownership were trending upward until 1990. One concern is that this trend is driving my results. I redo the regressions by limiting the sample to 1991-2010 and present the results in Table 5. Predictions of the model hold even after the strong growth in indexing was over.

5.3.5 Non-index stocks

While the previous results are consistent with my hypothesis, it could also be that all stocks in the size range of small S&P 500 stocks began to comove more with large S&P 500 stocks, contemporaneously with increases in passive ownership. In order to verify that this is an indexing effect, I form a sample consisting of the smallest 250 stocks in the S&P 500 and the largest 250 stocks in the non-S&P 500 portfolio. The size ratio of the bottom S&P 500 portfolio to the top non-S&P 500 portfolio varies between 0.51 and 1.18 with an average of 0.87. Market capitalization is not the only determinant for inclusion in the S&P 500. Other factors such as liquidity and market representation are also taken into consideration, which can exclude very large firms from the index. Probably the most notable example is Berkshire Hathaway Inc., which was only included in the S&P 500 (and S&P 100) in 2010.
Table 5

Results for regressions of changes in comovement estimates ($\Delta \beta_{SP500,t}$ and $\Delta \beta_{nonSP500,t}$) on the lagged changes in passive ownership ($\Delta PO$), from 1991 to 2010 (annual), and on $\Delta \beta_{t-1}$ which is the lagged value of the dependent variable in the regression, for S&P 500 firms. The dependent variable are the changes in average $\beta$s. Comovement $\beta$s are estimated from the regression $R_{j,t} = \alpha_j + \beta_{j,SP500,t}R_{SP500,t} + \beta_{j,nonSP500,t}R_{nonSP500,t} + u_{j,t}$, where $R_{SP500,t}$ is the value-weighted return of the S&P 500 stocks portfolio (excluding stock $j$) and $R_{non-SP500,t}$ is the value-weighted return of the rest of the market. Comovement $\beta$s are estimated based on index membership at the end of December, using daily data for the following 12 months. PO is the total passive assets under management divided by the total CRSP market capitalization and acts as a proxy for the relative wealth of indexers. T-stats are presented in parenthesis. I use heteroscedasticity-consistent standard errors. *, **, *** indicate significance at 10%, 5% and 1% level respectively.

<table>
<thead>
<tr>
<th>Dep. var.</th>
<th>$\Delta \beta_{SP500,t}$</th>
<th>$\Delta \beta_{nonSP500,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-0.01 (-0.53)</td>
<td>-0.01 (-0.28)</td>
</tr>
<tr>
<td>$\Delta PO_{t-1}$</td>
<td>10.00* (2.06)</td>
<td>9.61* (1.75)</td>
</tr>
<tr>
<td>$\Delta \beta_{t-1}$</td>
<td>-0.29 (-0.88)</td>
<td>-0.26 (-0.97)</td>
</tr>
<tr>
<td>Adj. $R^2$</td>
<td>0.17 0.04 0.20 0.19 -0.05</td>
<td>0.14</td>
</tr>
<tr>
<td>Obs.</td>
<td>20 20 20 20 20 20</td>
<td></td>
</tr>
</tbody>
</table>
The average comovement $\beta$s for the two groups are presented in Figure [10]. There is a clear difference between the time series patterns of the two groups.

I do a difference-in-difference analysis, in which the continuous treatment is the change in PO and the treated group ($T$) is the portfolio of small S&P 500 stocks. The control group is the portfolio of large non-index stocks. The hypothesis is that only the $\Delta \text{PO}_{t-1}$ for the treated group ($\Delta \text{PO}_{t-1} \times T$) should have significant coefficients with the expected signs. Results, shown in Table [6], support the hypothesis. In all regressions, the coefficient on $\Delta \text{PO}$ is small and not statistically significant. As expected, changes in passive ownership are not related to changes in comovement for non-index stocks. For the regressions on $\Delta \beta_{\text{SP500}}$ the coefficients on $\Delta \text{PO} \times T$ are of the expected sign and statistically significant. This suggests that passive ownership only affects comovement of index stocks. For the regressions on $\Delta \beta_{\text{nonSP500}}$ the coefficients on $\Delta \text{PO} \times T$ are of the expected sign but not statistically significant.

6 Concluding remarks

I present theoretical and empirical evidence suggesting that comovement within index stocks increases with the relative wealth of indexers. This can explain the rise of comovement in S&P 500 stocks that began in the 1970s. While the notion of excess comovement in index stocks is not new, this is the first paper to link it directly to the relative wealth of indexers and to its effect on discount rates.

I first introduce an asset-pricing model where the economy is populated by two rational representative agents, an active investor and an indexer. Two of the stocks form an index, and the indexer is constrained to invest solely in the bond and the index. The model predicts that, as the relative wealth of the indexer increases, index stocks comove more with other index stocks and less with non-index stocks. While this model proved useful for my analysis, I am aware of its limitations. One such limitation of the model is its inability to explain the rise of indexing since the model is not stationary; over time the active investor dominates. Future research could address this issue, introducing frictions such as the incremental cost of active investing, for which French (2008) provide estimates, and allowing one rational agent to invest both passively and actively at same time through delegated portfolio management, as in Petajisto (2009).

Second, I show that comovement of S&P 500 stocks with other S&P 500 stocks increases while comovement with non-S&P 500 stocks decreases as the share of passive holdings in the index increases. I show that these results are robust to some popular alternative explanations of comovement, including the correlated trading view.

following the 50-1 split of its Class B shares, making it the 21st largest company in the index.
Figure 10
Average annual comovement $\beta$s from 1957 to 2010 estimated using a $[+1, +12]$ months window from the bivariate regression $R_{j,t} = \alpha_j + \beta_{j,SP500}R_{SP500,t} + \beta_{j,nonSP500}R_{nonSP500,t} + u_{j,t}$. Results are based on index membership on the last trading day of the year, using returns from the following year.
Table 6

Results for difference-in-difference regressions with continuous treatment of changes in comovement estimates (\(\Delta \beta_{SP500,t}\) and \(\Delta \beta_{nonSP500,t}\)) on the lagged changes in incremental passive ownership (\(\Delta PO\)), from 1983 to 2010 (annual), and on \(\Delta \beta_{t-1}\) which is the lagged value of the dependent variable in the regression. The sample includes the smallest 250 stocks in the S&P 500 (the treated group \(T\)) and the largest 250 stock not in the S&P 500 (the control group). The regressions use the changes in average \(\beta\)s (within-group averages) as the dependent variable. Comovement \(\beta\)s are estimated from the regression:

\[
R_{j,t} = \alpha_j + \beta_j,SP500,t R_{SP500,t} + \beta_j,nonSP500,t R_{nonSP500,t} + u_{j,t},
\]

where \(R_{SP500,t}\) is the value-weighted return of the S&P 500 stocks portfolio (excluding stock \(j\) if \(j\) is in the index) and \(R_{non-SP500,t}\) is the value-weighted return of the rest of the market (excluding stock \(j\) if \(j\) is not in the index). Comovement \(\beta\)s are estimated based on index membership at the end of December, using daily data for the following 12 months. PO is the total passive assets under management divided by the total CRSP market capitalization and acts as a proxy for the relative wealth of indexers. T-stats are presented in parenthesis. I use heteroscedasticity-consistent standard errors. *, **, *** indicate significance at 10%, 5% and 1% level respectively.

<table>
<thead>
<tr>
<th>Dep. var.</th>
<th>(\Delta \beta_{SP500,t})</th>
<th>(\Delta \beta_{nonSP500,t})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(0.12)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>(T)</td>
<td>-0.03</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>(-0.92)</td>
<td>(-0.86)</td>
</tr>
<tr>
<td>(\Delta PO_{t-1})</td>
<td>-2.12</td>
<td>-1.45</td>
</tr>
<tr>
<td></td>
<td>(-0.61)</td>
<td>(-0.44)</td>
</tr>
<tr>
<td>(\Delta PO_{t-1} \times T)</td>
<td>13.81**</td>
<td>14.14**</td>
</tr>
<tr>
<td></td>
<td>(2.13)</td>
<td>(2.30)</td>
</tr>
<tr>
<td>(\Delta \beta_{t-1})</td>
<td>-0.35**</td>
<td>-0.07</td>
</tr>
<tr>
<td></td>
<td>(-2.64)</td>
<td>(-0.30)</td>
</tr>
<tr>
<td>Adj. (R^2)</td>
<td>0.07</td>
<td>0.18</td>
</tr>
<tr>
<td>Obs.</td>
<td>52</td>
<td>52</td>
</tr>
</tbody>
</table>
References


A Proofs and model derivation

A.1 Setup

This section provides additional details on the model presented in Section 3. The basket of stocks 1 and 2 is called the index $I$, which by construction represents a value-weighted index, and the basket of stocks 1, 2 and 3 is called the market $M$. The index and market baskets therefore also pay dividend streams with dynamics as described in (1), with the exception that their variance parameters have the form

$$\sigma_{D_I} = \left[1 - \frac{2s_1s_2}{(s_1 + s_2)^2} (1 - \rho_D) \right] \sigma_D^2,$$

$$\sigma_{D_M} = \left[1 - 2(s_1s_2 + s_1s_3 + s_2s_3)(1 - \rho_D) \right] \sigma_D^2,$$

where $s_i$ is the weight of share of dividends of asset $i$, i.e.

$$s_i = \frac{D_i}{D_1 + D_2 + D_3}, \quad i \in \{1, 2, 3\}.$$

Let $\omega_{i,t}$ denote the market weight of stock $i$ at time $t$ such that $\sum_{i=1}^3 \omega_{i,t} = 1$ and let $\omega_{i,t}^l = \omega_{i,t}/(\omega_{1,t} + \omega_{2,t})$ denote the weight of asset $i \in \{1, 2\}$ in the index. Then the index return moments are

$$\mu_{I,t} = \omega_{I,1}^l \mu_{1,t} + \omega_{I,2}^l \mu_{2,t},$$

$$\sigma_{I,t}^2 = (\omega_{I,1}^l)^2 \sigma_{1,t}^2 + (\omega_{I,2}^l)^2 \sigma_{2,t}^2 + 2\omega_{I,1}^l \omega_{I,2}^l \text{corr}(dZ_{1,t}, dZ_{2,t}) \sigma_{1,t} \sigma_{2,t}.$$

A.2 Agents’ problem

Agent $j$’s optimization problem at time $t$ is to maximizes her time additive utility

$$U_{j,t} = E_t \left[ \int_t^{\infty} e^{-\delta(s-t)} \log c_{j,s} ds \right]$$

subject to her budget constraint. Formally, this gives

$$\max U_{j,t} \text{ subject to } E_t \left[ \int_0^{\infty} \frac{\xi_{j,s}}{\xi_{j,t}} c_{j,s} ds \right] \leq W_{j,t},$$

where $\xi_{j,t}$ is the marginal utility of agent $j$ at time $t$. The first order condition is

$$\kappa_j \frac{\xi_{j,s}}{\xi_{j,t}} = e^{-\delta(s-t)} c_{j,s}^{-1}.$$
where \( \kappa_j \) is the Lagrange multiplier on the budget constraint and \( \xi_{j,t} \) is a process given by
\[
\frac{d\xi_{j,t}}{\xi_{j,t}} = -r_{j,t} dt - \theta'_{j,t} dZ_t.
\] (36)

where \( \theta_{j,t} \) is the price of risk process for agent \( j \). Note that the process can also be written with respect to the dividend basis and the market basis as
\[
\frac{d\xi_{j,t}}{\xi_{j,t}} = -r_{j,t} dt - \theta'_{j,t} dZ^D_{D,t} = -r_{j,t} dt - \theta'_{j,t} dZ_t.
\] (37)

The rationale for using two different bases, in addition to the initial Brownian motions \( Z \), is that each of the two new bases simplifies the solution for a part of the problem and involves independent Brownian motions, which are easier to deal with. It is simpler to solve for optimal portfolios and market clearing under the market basis. However, the market basis transformation depends on stock return covariances, so it is not appropriate to solve for equilibrium price dynamics. The dividend basis is much useful for that purpose.

Since both agents trade in the bond, in equilibrium they should have the same riskless rate (i.e. \( r_{\mathcal{I},t} = r_{A,t} = r_t \)). However their different investment opportunity sets means they will face different market price of risk. Following the convex duality methodology approach of \cite{Cvitanić and Karatzas 1992}, I define a fictitious market which the indexer views as complete. In the current setup with log utility, the market price of risk in the fictitious market is the same as in the incomplete market (See Example 7.2 on p.304 \cite{Karatzas and Shreve 1998} for more details.)

The idea is to create a fictitious market for agent \( \mathcal{I} \) by replacing the expected return on asset \( i \) by \( \mu_i(\psi) = \mu_i + \psi_i \) such that in equilibrium she chooses not to hold the unavailable asset, and to hold the index assets according to index weights. In the present setup,
\[
\psi = \arg\min_{\psi} \left[ (\mu_1(\psi) - r, \mu_2(\psi) - r, \mu_3(\psi) - r) \Sigma^{-1} (\mu_1(\psi) - r, \mu_2(\psi) - r, \mu_3(\psi) - r)^\prime \right]^{1/2}.
\] (38)

Substituting the \( \psi \) obtained in (38) in the shadow market price of risk of the indexer I obtain, under the market basis,
\[
\theta_{\mathcal{I}} = \phi_I \sigma_I^{-1} \begin{bmatrix} \sigma_1 \omega_1^I + \rho_{12} \sigma_2 \omega_2^I & \rho_{12} \sigma_2 \omega_2^I \sqrt{1 - \rho_{12}^2 \sigma_2^I} \\ \sqrt{1 - \rho_{12}^2 \sigma_2^I} & 0 \end{bmatrix}.
\] (39)

where \( \phi_I = \frac{\mu_I - r}{\sigma_I} \) is the Sharpe ratio of the index. Since \( (\sigma_1 \omega_1^I + \rho_{12} \sigma_2 \omega_2^I)^2 + (\sqrt{1 - \rho_{12}^2 \sigma_2^I})^2 = \sigma_I^2 \), in scalar form \( \theta_{\mathcal{I}} = \phi_I \). The result in (39) has the same form if working under the dividend basis.

\[20\] For a definition of the different bases, see Appendix A.11.
basis following (37):

\[
\bar{\theta}_t = \phi_{1} \sigma_t^{-1} \begin{bmatrix}
\omega_1 \sigma_{11} + \omega_2 \sigma_{21}
\omega_1 \sigma_{12} + \omega_2 \sigma_{22}
\omega_1 \sigma_{13} + \omega_2 \sigma_{23}
\end{bmatrix},
\]

(40)

Agent \( A \) is unconstrained and faces complete markets, so her market price of risk under the market and dividend bases are given by

\[
\theta_A = \sigma^{-1}(\mu_1 - r, \mu_2 - r, \mu_3 - r)'
\]

\[
= \begin{bmatrix}
\phi_1 \\
\phi_2 (1 - \rho_1 \rho_2) \sqrt{1 - \rho_{12}^2} \\
\phi_3 (1 - \rho_1 \rho_3) \sqrt{1 - \rho_{13}^2} - \phi_3 (1 - \rho_1 \rho_3 - \rho_2 \rho_3) - \phi_1 (1 - \rho_1 \rho_2) \sqrt{1 - \rho_{12}^2}
\end{bmatrix}
\]

\[
\bar{\theta}_A = \sigma^{-1}(\mu_1 - r, \mu_2 - r, \mu_3 - r)'
\]

\[
= \frac{1}{c} \begin{bmatrix}
x_1 (\sigma_{12} \sigma_{32} - \sigma_{22} \sigma_{32}) + x_2 (\sigma_{12} \sigma_{33} - \sigma_{22} \sigma_{33}) + x_3 (\sigma_{13} \sigma_{22} - \sigma_{12} \sigma_{32}) \\
x_1 (\sigma_{21} \sigma_{33} - \sigma_{23} \sigma_{31}) + x_2 (\sigma_{13} \sigma_{31} - \sigma_{11} \sigma_{33}) + (x_3 \sigma_{11} \sigma_{23} - \sigma_{13} \sigma_{21}) \\
x_1 (\sigma_{22} \sigma_{31} - \sigma_{21} \sigma_{32}) + x_2 (\sigma_{12} \sigma_{32} - \sigma_{12} \sigma_{31}) + x_3 (\sigma_{12} \sigma_{21} - \sigma_{11} \sigma_{22})
\end{bmatrix},
\]

(42)

where

\[
c = \sigma_{13} (\sigma_{22} \sigma_{31} - \sigma_{21} \sigma_{32}) + \sigma_{12} (\sigma_{21} \sigma_{33} - \sigma_{23} \sigma_{31}) + \sigma_{11} (\sigma_{23} \sigma_{32} - \sigma_{22} \sigma_{33})
\]

and \( x_i = \mu_i - r \) is the excess return on asset \( i \).

### A.3 Optimal portfolios

Agent \( A \) is unconstrained, so her optimal portfolio proportions are given by

\[
\pi_{A,t} = \Sigma_t^{-1}(\mu_t - r \mathbf{1}).
\]

(43)

Under the market basis the covariance matrix is \( \Sigma_t = \sigma_t \sigma_t' \), so

\[
\pi_A = \begin{bmatrix}
\phi_1 (1 - \rho_{12}^2) - \phi_2 (\rho_{12} - \rho_{13} \rho_{23}) - \phi_3 (\rho_{13} - \rho_{12} \rho_{23}) \\
\sigma_1 (1 - \rho_{12}^2 - \rho_{13}^2 + 2 \rho_{12} \rho_{13} \rho_{23}) \\
\phi_2 (1 - \rho_{12}^2 - \phi_1 (\rho_{12} - \rho_{13} \rho_{23}) - \phi_3 (\rho_{13} - \rho_{12} \rho_{23}) \\
\sigma_2 (1 - \rho_{12}^2 - \rho_{13}^2 + 2 \rho_{12} \rho_{13} \rho_{23}) \\
\phi_3 (1 - \rho_{12}^2) - \phi_1 (\rho_{12} - \rho_{13} \rho_{23}) - \phi_2 (\rho_{13} - \rho_{12} \rho_{23}) \\
\sigma_3 (1 - \rho_{12}^2 - \rho_{13}^2 + 2 \rho_{12} \rho_{13} \rho_{23})
\end{bmatrix}.
\]

(44)
As for agent $I$, I know from Cvitanić and Karatzas (1992) that $\pi_{I,t}$ coincides with the optimal portfolio in the incomplete market:

$$\pi_I = \begin{bmatrix} \frac{\pi_I^1}{\pi_I^2} \\ \frac{\pi_I^2}{\pi_I^2} \\ 0 \end{bmatrix},$$  \hspace{1cm} (45)

where $\pi_{I,t}^I = (\mu_{I,t} - r)/\sigma_{I,t}^2$, so

$$\pi_I = \begin{bmatrix} \omega_I^1 \phi_I^1 \\ \omega_I^2 \phi_I^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_I^1 (x_1 \omega_I^1 + x_2 \omega_I^2) \\ \frac{(x_1 \omega_I^1 + x_2 \omega_I^2)^2 + 2p_1 \sigma_1 \sigma_2 \omega_I^1 \omega_I^2 + \sigma_2^2 \omega_I^2}{\sigma_1^2 \omega_I^1 + 2p_1 \sigma_1 \sigma_2 \omega_I^1 \omega_I^2 + \sigma_2^2 \omega_I^2} \\ 0 \end{bmatrix}.$$  \hspace{1cm} (46)

**A.4 Proof of Proposition [1]**

The market clearing condition imposes that

$$\omega_t = \pi_{A,t} \nu_{A,t} + \pi_{I,t} \nu_{I,t},$$

so

$$\omega_t = \begin{bmatrix} \nu_A (x_3 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_2 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_1 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2) \\ \nu_A (x_3 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_2 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_1 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2) \end{bmatrix} \begin{bmatrix} \omega_I^1 (x_1 \omega_I^1 + x_2 \omega_I^2) \\ \omega_I^2 (x_1 \omega_I^1 + x_2 \omega_I^2) \end{bmatrix}.$$  \hspace{1cm} (47)

where $x_i = \mu_i - r$ are excess returns. Solving for $x_1$, $x_2$ and $x_3$, I get

$$x_1 = (\sigma_1 (\sigma_2 \sigma_3 \rho_2 \omega_3 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_2 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_1 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2))$$

$$+ \frac{\nu_A (x_3 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_2 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2 + x_1 (\rho_{12} \rho_{13} \rho_{23} - 1) \sigma_1 \sigma_2)}{\nu_A (\sigma_1^2 \omega_1^1 + 2p_1 \sigma_1 \sigma_2 \omega_1 \omega_2 + \sigma_2^2 \omega_2)} \left(1 - \frac{\omega_I^1 (\omega_1 \rho_{13} \sigma_1 + \omega_2 \rho_{23} \sigma_2)}{\sigma_I^2}ight),$$

$$+ \frac{\omega_2 \omega_3 \sigma_3 \sigma_3}{\nu_A \sigma_I^2} \left[\omega_1 \sigma_1 (\rho_{12} \rho_{13} - \rho_{23}) - \omega_2 \sigma_2 (\rho_{12} \rho_{23} - \rho_{13})\right].$$  \hspace{1cm} (48)

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I can also write $x_1^*$ in terms of $x_I^*$:

$$x_1^* = \frac{1}{\nu_A \sigma_I^2 \omega^*_I} \left\{ \sigma_1 (\sigma_2 \sigma_3 \omega_3 \omega_1 \omega_2 - 1) + \rho_1 \sigma_1 \omega_1 + \rho_2 \sigma_2 \omega_2 \right\}$$

$$+ \frac{1}{\sigma_I^2 \omega^*_I} \left\{ x_I^* (\sigma_1 \omega_1 + \rho_1 \sigma_2 \omega_2) \right\}$$

$$= \frac{1}{\sigma_I^2 \omega^*_I} \left\{ x_I^* (\sigma_1 \omega_1 + \sigma_2 \sigma_2 \omega_2) \right\}$$

$$+ \frac{1}{\nu_A} \left( \frac{\omega_1}{\omega_I} \left[ \text{cov}(R_1, R_2) \text{cov}(R_1, R_3) - \sigma_1^2 \text{cov}(R_2, R_3) \right] \right)$$

$$- \frac{\omega_2}{\omega_I} \left[ \text{cov}(R_1, R_2) \text{cov}(R_2, R_3) - \sigma_2^2 \text{cov}(R_1, R_3) \right] \right\}.$$

For $x_3^*$, I get

$$x_3^* = \left\{ \sigma_3 \left( \nu_A \rho_1 \sigma_1^2 \omega_3 + \sigma_1^2 \omega_1^2 \left[ \nu_A \left( 2 \rho_1 \rho_2 \sigma_2 \omega_2 + \rho_3 \right) \right] \right) + \left( 1 + (-1 + \nu_A) \rho_1 \right) \sigma_3 \omega_3 \right\}$$

$$= \frac{\omega_I \text{cov}(R_1, R_3) + \omega_3 \sigma_3^2 \left[ 1 + \frac{\nu_I}{\sigma_3^2} (1 - \rho_3) \right]}{\omega_1 + \omega_2}.$$

where

$$x_I^* = \frac{\sigma_1^2 \omega_1^2 + 2 \rho_1 \sigma_1 \omega_1 \omega_2 + \sigma_2^2 \omega_2^2 + \rho_1 \sigma_1 \sigma_3 \omega_1 \omega_3 + \rho_2 \sigma_2 \sigma_3 \omega_2 \omega_3}{\omega_1 + \omega_2} = \sigma_1^2 \omega_1 + \omega_3 \text{cov}(R_1, R_3),$$

with $\omega_I = \omega_1 + \omega_2$. Results for $x_2$ are omitted as they are symmetric to $x_1$.

**A.5 Proof of Proposition 2**

Following Cuoco and He (1994), I can still use a social planner to derive equilibrium prices, but the weight $\lambda_t$ will be stochastic:

$$U_t = E_t \int_t^\infty e^{-\delta(s-t)} \left( \log c_{A,s} + \lambda_s \log c_{T,s} \right) ds.$$

The consumption sharing rule is given by

$$1 = \frac{c_{A,t}^{-1}}{\lambda_t c_{T,t}^{-1}}.$$
I define Agent $j$’s equilibrium share of world consumption as $\nu_{j,t} = \frac{c_{j,t}}{D_{M,t}}$. In equilibrium the two agents must consume the aggregate dividend: $c_{A,t} + c_{I,t} = D_{M,t}$. Thus

$$\nu_{A,t} = \frac{1}{1 + \lambda_t}, \quad \nu_{I,t} = \frac{\lambda_t}{1 + \lambda_t}. \quad (54)$$

As in Basak and Cuoco (1998), the equilibrium state-price density $\xi_t$ is given by the state-price density of the unconstrained agent $A$:

$$\xi_t = \xi_{A,t} = \kappa_A e^{-\delta t}(\nu_{A,t}D_{M,t})^{-1}. \quad (55)$$

To solve for equilibrium prices, I need to derive an expression $\lambda_t$ and the related process $\nu_{A,t}$. Substituting $c_A$ and $c_I$ from (35) in (53), I get

$$\lambda_t = \frac{\kappa_A \xi_{A,t}/\xi_{A,0}}{\kappa_I \xi_{I,t}/\xi_{I,0}}. \quad (56)$$

Solving (37), agent $j$’s state-price density under the dividend basis, gives:

$$\xi_{j,t} = \xi_{j,0}e^{-\int_0^t (r_s + \frac{1}{2} \theta^2_{j,s})ds - \int_0^t \bar{\theta}_{j,s}d\bar{Z}_{D,s}} \quad (57)$$

where $\theta_{j,s} = \bar{\theta}_{j,s} \mathbf{1}$ and $\mathbf{1}$ is a vector of ones. Substitution (57) in (56) gives

$$\lambda_t = \frac{\kappa_A e^{-\int_0^t \frac{1}{2}(\theta^2_{A,s} - \theta^2_{I,s})ds - \int_0^t (\bar{\theta}_{A,s} - \bar{\theta}_{I,s})d\bar{Z}_{D,s}}}{\kappa_I \xi_{I,t}/\xi_{I,0}}. \quad (58)$$

Applying Itô’s Lemma gives

$$\frac{d\lambda_t}{\lambda_t} = \mu_{\lambda,t}dt + \sigma_{\lambda,t}d\bar{Z}_{D,t} \quad (59)$$

where

$$\mu_{\lambda,t} = \bar{\theta}_{I,t}(\bar{\theta}_{I,t} - \bar{\theta}_{A,t}), \quad (60)$$

$$\sigma_{\lambda,t} = (\bar{\theta}_{I,t} - \bar{\theta}_{A,t}). \quad (61)$$

Rewriting as a scalar process,

$$\frac{d\lambda_t}{\lambda_t} = \mu_{\lambda,t}dt + \sigma_{\lambda,t}dZ_{\lambda,t} \quad (62)$$

where

$$\sigma_{\lambda,t} = \sqrt{(\bar{\theta}_{I,t} - \bar{\theta}_{A,t})(\bar{\theta}_{I,t} - \bar{\theta}_{A,t})} \quad (63)$$

$$dZ_{\lambda,t} = \sigma_{\lambda,t}^{-1}(\bar{\theta}_{I,t} - \bar{\theta}_{A,t})d\bar{Z}_{D,t}. \quad (64)$$
Remember that
\[
\theta_I = \frac{x_I}{\sigma_I} \sigma' \begin{bmatrix}
\begin{array}{c}
\omega_1' \\
\omega_2' \\
0
\end{array}
\end{bmatrix},
\quad \theta_A = \sigma^{-1} \begin{bmatrix}
\begin{array}{c}
x_1 \\
x_2 \\
x_3
\end{array}
\end{bmatrix}.
\]

Therefore,
\[
\theta_I - \theta_A = \frac{x_I}{\sigma_I} \sigma' \begin{bmatrix}
\begin{array}{c}
\omega_1' \\
\omega_2' \\
0
\end{array}
\end{bmatrix} - \sigma^{-1} \begin{bmatrix}
\begin{array}{c}
x_1 \\
x_2 \\
x_3
\end{array}
\end{bmatrix}
\]

where \(\beta_{I,3} = \rho_{I,3} \sigma_3 / \sigma_I = (\omega_1' \rho_{13} \sigma_1 \sigma_3 + \omega_2' \rho_{23} \sigma_2 \sigma_3) / \sigma_I^2\). One can easily see that \(\theta_I' \theta_I = \frac{x_I^2}{\sigma_I^2}\) and that \(\theta_I' \theta_A = \frac{x_I^2}{\sigma_I^2}\). Note that those results are basis invariant. I obtain
\[
\mu_\lambda = \theta_I' (\theta_I - \theta_A) = 0.
\]

Similarly,
\[
\sigma_\lambda^2 = (\theta_I - \theta_A)' (\theta_I - \theta_A)
=
-\frac{x_I^2}{\sigma_I^2} + \theta_A' \theta_A.
\]
\[
\Rightarrow \sigma_\lambda = \sqrt{[x_1 \ x_2 \ x_3] \Sigma^{-1} [x_1 \ x_2 \ x_3]' - \frac{x_I^2}{\sigma_I^2}}.
\]

Using the definition of \(\nu_A\) in (54) and applying Itô Lemma gives
\[
d\nu_A = \mu_{\nu_A} dt + \sigma_{\nu_A}' d\mathcal{Z}_D
\]

where
\[
\mu_{\nu_A} = \nu_A \sigma_{\lambda}^2 \nu_A^2,
\quad \sigma_{\nu_A} = \nu_A \sigma_I \sigma_{\lambda}.
\]
In scalar notation this becomes

\[ d\nu_A = \mu_{\nu_A} dt + \sigma_{\nu_A} dZ, \]  
\[ \sigma_{\nu_A} = -\nu_A \nu_I \sigma \lambda. \]  

(73)  

(74)

Applying Itô’s Lemma to (55), I obtain

\[ \frac{d\xi}{\xi} = -\left[ \delta + \mu_{DM} - \sigma_{DM}^2 + \frac{\rho_{\nu_AM} \sigma_{\nu_A} \sigma_{DM}}{\nu_A} \right] dt \]

\[ - \left[ \frac{\sigma_{DM}}{\nu_A} + \frac{\sigma'_{\nu_A}}{\nu_A} \right] dZ_D \]  

(75)

Equalling the terms to those in (37),

\[ r_f = \delta + \mu_{DM} - \sigma_{DM}^2 + \frac{\rho_{\nu_AM} \sigma_{\nu_A} \sigma_{DM}}{\nu_A}, \]

\[ \theta = \sigma_{DM} + \frac{\sigma'_{\nu_A}}{\nu_A}. \]  

(76)  

(77)

A.6 Proof of Corollary

From (55) I can assert that \( \overline{\theta} = \overline{\theta}_A \). Thus, from (61), (72) and (77),

\[ \overline{\theta}_A = \sigma_{DM} + \frac{\sigma'_{\nu_A}}{\nu_A} \]

\[ = \sigma_{DM} - \nu_I \sigma \lambda \]

\[ = \sigma_{DM} - \nu_I (\overline{\sigma}_I - \overline{\theta}_A), \]  

(78)  

\[ \Rightarrow \overline{\theta} = \frac{\sigma_{DM}}{\nu_A} - \frac{\nu_I}{\nu_A} \overline{\theta}_I, \]  

(79)  

\[ \overline{\theta} = \sigma_{DM} + \nu_I \left( \frac{\sigma_{DM}}{\nu_A} - \overline{\theta}_I \right). \]  

(80)

Note here that \( \sigma_{DM} \) is exogenous to the model (when defined relative to the dividend basis), \( \nu_I \) and \( \nu_A = 1 - \nu_I \) are state variables and the other quantities are determined endogenously in equilibrium. Denoting \( \overline{\theta}^* = \sigma_{DM} \), the price of risk when there are no indexers (\( \nu_A = 1, \nu_I = 0 \)),

\[ \overline{\theta} = \overline{\theta}^* + \frac{\nu_I}{\nu_A} \left( \overline{\theta}^* - \overline{\theta}_I \right) \]

\[ \Rightarrow \left( \overline{\theta}^* - \overline{\theta}_A \right) = -\frac{\nu_I}{\nu_A} \left( \overline{\theta}^* - \overline{\theta}_I \right). \]  

(81)
A.7 Proof of Proposition 3

In this section I derive the dynamics of each stock’s price process. The price $S_{i,t}$ of stock $i$ at time $t$ is the expected value of future dividends discounted using the stochastic discount factor of the representative agent $\xi$ defined in (75):

$$S_{i,t} = E_t \left[ \int_t^\infty \frac{\xi_\tau}{\xi_t} D_{i,\tau} d\tau \right]$$

(82)

Using the results from equations (35) and (55), I have

$$S_{i,t} = E_t \left[ \int_t^\infty e^{-\delta(\tau-t)} \left( \frac{c_{A,\tau}}{c_{A,t}} \right)^{-1} D_{i,\tau} d\tau \right]$$

(83)

From (54), I have

$$c_{A,t} = \frac{D_{M,t}}{1 + \lambda_t},$$

(84)

thus

$$\frac{c_{A,\tau}}{c_{A,t}} = \frac{D_{M,\tau}}{D_{M,t}} \frac{1 + \lambda_t}{1 + \lambda_\tau}.$$  

(85)

Substituting this last result in (83), I obtain

$$S_{i,t} = D_{M,t} E_t \left[ \int_t^\infty e^{-\delta(\tau-t)} \frac{1 + \lambda_\tau}{1 + \lambda_t} s_{i,\tau} d\tau \right]$$

$$= D_{M,t} f_{i,t},$$  

(86)

where

$$f_{i,t} = E_t \left[ \int_t^\infty e^{-\delta(\tau-t)} \frac{1 + \lambda_\tau}{1 + \lambda_t} s_{i,\tau} d\tau \right]$$

$$= \frac{1}{1 + \frac{\lambda_t}{\nu_{A,t}}} E_t \left[ \int_t^\infty e^{-\delta(\tau-t)} s_{i,\tau} d\tau \right] + \frac{\lambda_t}{1 + \frac{\lambda_t}{\nu_{I,t}}} E_t \left[ \int_t^\infty e^{-\delta(\tau-t)} \frac{\lambda_\tau}{\lambda_t} s_{i,\tau} d\tau \right]$$

$$= \nu_{A,t} f^A_{i,t} + \nu_{I,t} f^I_{i,t}.$$  

(87)

$$f^A_{i,t} = \frac{1}{1 + \frac{\lambda_t}{\nu_{A,t}}} E_t \left[ \int_t^\infty e^{-\delta(\tau-t)} s_{i,\tau} d\tau \right]$$

$$f^I_{i,t} = \frac{\lambda_t}{1 + \frac{\lambda_t}{\nu_{I,t}}} E_t \left[ \int_t^\infty e^{-\delta(\tau-t)} \frac{\lambda_\tau}{\lambda_t} s_{i,\tau} d\tau \right].$$

(88)

Note that in a world without constraints, $\lambda_t$ is constant and we thus have $f_{i,t} = f^A_{i,t}$. Alternatively, I can get this result by setting $\nu_{A,t} = 1$ and $\nu_{I,t} = 0$. 

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A.8 Solving for $f_{i,t}^A$

$f_{i,t}^A$ depends on the relative share of the aggregate dividend of each stock, $s_{i,t}$ as defined in (30). Therefore,

$$s_{i,t} = \frac{D_{i,t}}{D_{M,t}}. \quad (89)$$

To fully characterize the relative weights of each dividend stream, two of those $s_i$ are sufficient, so I need two state variables. Using Itô’s Lemma, I obtain

$$\frac{ds_i^M}{s_i^M} = \left[ \sigma'_M (\sigma_M - \sigma_{D_i}) \right] dt$$

$$+ (\sigma_{D_i} - \sigma_{D_M})' dZ_D, \quad (90)$$

which after simplification yields

$$ds_i = \mu_{s_i} dt + \sigma'_{s_i} dZ_D, \quad (91)$$

where

$$\mu_{s_i} = s_i s_{-i} \left[ -s_i \sigma_D^2 + s_{-i} \sigma_{D_{-i}}^2 + (s_i - s_{-i}) \rho_{D_i D_{-i}} \sigma_D \sigma_{D_{-i}} \right] \quad (92)$$

$$\sigma_{s_i} = s_i s_{-i} (\sigma_{D_i} - \sigma_{D_{-i}}), \quad (93)$$

and $D_{-i}$ represents the dividend stream of the other two stocks combined.

Defining $x_{i,t} = \log \frac{s_{i,t}}{s_{-i,t}}$, it follows from Itô’s Lemma that:

$$dx_i = \mu_{x_i} dt + \sigma'_{x_i} dZ_D \quad (94)$$

where

$$\mu_{x_i} = \left[ \mu_{D_i} - \frac{1}{2} \sigma_{D_i}^2 \right] - \left[ \mu_{D_{-i}} - \frac{1}{2} \sigma_{D_{-i}}^2 \right], \quad (95)$$

$$\sigma_{x_i} = \sigma_{D_i} - \sigma_{D_{-i}}. \quad (96)$$

In scalar form,

$$dx_i = \mu_{x_i} dt + \sigma_{x_i} dZ_{x_i} \quad (97)$$

where

$$\sigma_{x_i} = \sqrt{(\sigma_{D_i} - \sigma_{D_{-i}})'(\sigma_{D_i} - \sigma_{D_{-i}})}$$

$$= \sqrt{\sigma_{D_i}^2 + \sigma_{D_{-i}}^2 - 2 \rho_{D_i D_{-i}} \sigma_D \sigma_{D_{-i}}}, \quad (98)$$

$$Z_{x_i} = \sigma_{x_i}^{-1} \sigma'_{x_i} dZ_D. \quad (99)$$

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From Cochrane, Longstaff, and Santa-Clara (2008), I know there is a closed-form expression for $f_{A,i,t}$ if $x_i$ is the only relevant state variable ($\nu_A$ is irrelevant for $f_{A,i,t}$). In the present case the moments of the dividend process of portfolio $-i$ also depend on the relative dividend of the two stocks in that portfolio, i.e. $x_1$ depends on $D_2/D_3$. So $f_{A,i,t}$ depends on two state variables representing the relative dividend processes. Let’s use $x_1$ and $x_2$ as the state variables. Note also that since $\sum_{i=1}^3 f_{A,i,t} = \frac{1}{3}$, we only need to solve for two $i$ to get the third one. We’ll solve for $i = 1, 2$ so the functions will be symmetric. Here I show the derivation of $f_{A,1,t}$. Note from (91) that $s_i = 0$ and $s_i = 1$ are absorbing states, so we obtain the following conditions:

$$\lim_{x_1 \to -\infty} f_{A,1,t} = 0,$$  \hspace{1cm} (100)

$$\lim_{x_1 \to \infty} f_{A,1,t} = \frac{1}{3},$$  \hspace{1cm} (101)

$$\lim_{x_2 \to \infty} f_{A,1,t} = 0.$$  \hspace{1cm} (102)

The boundary condition $\lim_{x_2 \to -\infty} f_{A,1,t}$ is less obvious because in that case asset 2 becomes irrelevant, so $f_{A,1,t}$ converges to the Cochrane, Longstaff, and Santa-Clara (2008) case. From the Feynman-Kac theorem,

$$\frac{1}{2} \sigma_{x_1}^2 \frac{\partial^2 f_{A,1,t}}{\partial x_1^2} + \frac{1}{2} \sigma_{x_2}^2 \frac{\partial^2 f_{A,1,t}}{\partial x_2^2} + \sigma_{x_1} \sigma_{x_2} \frac{\partial^2 f_{A,1,t}}{\partial x_1 \partial x_2} + \mu_{x_1} \frac{\partial f_{A,1,t}}{\partial x_1} + \mu_{x_2} \frac{\partial f_{A,1,t}}{\partial x_2} - \rho f_{A,1,t} + \frac{1}{1 + e^{-x_1}} = 0,$$  \hspace{1cm} (103)

where

$$\mu_{x_1} = -\left[ \frac{s_2 - s_1 s_2 - s_2^2}{(1 - s_1)^2} \right] (1 - \rho_D) \sigma_D^2$$

$$\mu_{x_2} = -\left[ \frac{s_1 - s_1 s_2 - s_1^2}{(1 - s_2)^2} \right] (1 - \rho_D) \sigma_D^2$$

$$\sigma_{x_1}^2 = \left[ 2 - \frac{2(s_2 - s_1 s_2 - s_2^2)}{(1 - s_1)^2} \right] (1 - \rho_D) \sigma_D^2$$

$$\sigma_{x_2}^2 = \left[ 2 - \frac{2(s_1 - s_1 s_2 - s_1^2)}{(1 - s_2)^2} \right] (1 - \rho_D) \sigma_D^2$$

$$\sigma_{x_1} \sigma_{x_2} = \left[ \frac{1 + s_1 (-3 + 2s_1) - 3s_2 + 2 s_1 s_2 + 2s_2^2}{(1 - s_1)(1 - s_2)} \right] (1 - \rho_D) \sigma_D^2.$$  \hspace{1cm} (103)

Following Bhamra (2007), I use a perturbation expansion of the form

$$f_{A,1} = f_{A,1,0} + \epsilon f_{A,1,1} + \epsilon^2 f_{A,1,2} + \ldots$$  \hspace{1cm} (104)
Defining $\rho_D = 1 - 2\epsilon^2$, I get

\begin{align*}
f_{1,0}^A &= \frac{1}{\delta + e^{-x_1}\delta}, \\
f_{1,1}^A &= 0, \\
f_{1,2}^A &= e^{x_1}(1 - e^{x_1}(-1 + s_1)^2 + s_1^2 + 2s_1(-1 + s_2) + 2(-1 + s_2)s_2)\sigma_D^2, \\
f_{1,3}^A &= 0.
\end{align*}

After simplification, I obtain

\[ f_{1}^A = \frac{s_1}{\delta} - \frac{s_1(1 - 3s_1 + 2s_1^2 - 2s_2 + 2s_1s_2 + 2s_2^2)(-1 + \rho_D)\sigma_D^2}{2\delta^2} + O(\epsilon^4). \tag{105} \]

### A.9 Solving for $f_{i,t}^T$

Remember that

\[ f_{i,t}^T = E_t\left[ \int_t^\infty e^{-\delta(\tau-t)}\frac{\lambda_\tau}{\lambda_t} s_{i,\tau} d\tau \right], \tag{106} \]

which depends on $x_{1,t}$, $x_{2,t}$ and $\nu_{A,t} = \frac{1}{1 + \lambda_t}$. Note that $\lambda$ is a local martingale and that assuming $\sigma_\lambda$ is bounded, then it is an exponential martingale. I can then define a new measure \[ P'(A_T) = E_t[1_{A_T}\lambda_T], \quad \forall t, \quad T \in [0, \infty) \quad t \leq T. \tag{107} \]

With this change of measure,

\[ f_{i,t}^T = E_t^{P'}\left[ \int_t^\infty e^{-\delta(\tau-t)}s_{i,\tau} d\tau \right]. \tag{108} \]

From (108), it follows that $f_{i,t}^T$ satisfies a BSDE. The coefficients of the BSDE will depend on $\nu_{i,t}^A$, which satisfies a FSDE. Together they form a FBSDE. The Feynman-Kac theorem still applies thus $f_{i,t}^T$ satisfies the following inhomogeneous elliptic PDE:

\begin{align*}
\mu_x^\nu \frac{\partial f_1^T}{\partial x_1} + \mu_x^\nu \frac{\partial f_1^T}{\partial x_2} + \mu_{\nu A} \frac{\partial f_1^T}{\partial \nu A} + \frac{1}{2} \sigma_{x_1}^2 \frac{\partial^2 f_1^T}{\partial x_1^2} + \frac{1}{2} \sigma_{x_2}^2 \frac{\partial^2 f_1^T}{\partial x_2^2} + \frac{1}{2} \sigma_{\nu A}^2 \frac{\partial^2 f_1^T}{\partial \nu A^2} \\
+ \sigma_{x_1} \sigma_{x_2} \frac{\partial^2 f_1^T}{\partial x_1 \partial x_2} + \sigma_{x_1} \sigma_{\nu A} \frac{\partial^2 f_1^T}{\partial x_1 \partial \nu A} + \sigma_{x_2} \sigma_{\nu A} \frac{\partial^2 f_1^T}{\partial x_2 \partial \nu A} - \rho f_1^T + \frac{1}{1 + e^{-x_1}} = 0, \tag{109}
\end{align*}

\[ ^{21}\text{See pages 28-29 of Karatzas and Shreve (1998) for details.} \]
where

\[
\begin{align*}
\mu'_{x_1} &= \mu_{x_1} + \sigma'_{x_1} \sigma_{\lambda}, \\
\mu'_{x_2} &= \mu_{x_2} + \sigma'_{x_2} \sigma_{\lambda}, \\
\mu'_{\nu_A} &= \mu_{\nu_A} + \sigma'_{\nu_A} \sigma_{\lambda} \quad = -\nu_A^2 (1 - \nu_A) \sigma_{\lambda}^2, \\
\sigma'_{x_1} \nu_A &= -\nu_A (1 - \nu_A) \sigma'_{x_1} \sigma_{\lambda}, \\
\sigma'_{x_2} \nu_A &= -\nu_A (1 - \nu_A) \sigma'_{x_2} \sigma_{\lambda}, \\
\sigma'_{x_1} \lambda &= \sigma'_{D_1} \sigma_{\lambda} - \left( \frac{s_2}{1 - s_1} \right) \sigma'_{D_2} \sigma_{\lambda} - \left( 1 - \frac{s_2}{1 - s_1} \right) \sigma'_{D_3} \sigma_{\lambda}, \\
\sigma'_{x_2} \lambda &= \sigma'_{D_2} \sigma_{\lambda} - \left( \frac{s_1}{1 - s_2} \right) \sigma'_{D_1} \sigma_{\lambda} - \left( 1 - \frac{s_1}{1 - s_2} \right) \sigma'_{D_3} \sigma_{\lambda}.
\end{align*}
\]

Note that \( \sigma_{\nu_A}^2 \) also depends on \( \sigma_{\lambda}^2 \) and that \( \sigma_{\lambda} \) (and \( \sigma_{\lambda}^2 \)) depends on the endogenously determined \( \sigma \).

### A.9.1 Boundary conditions

The required boundary conditions are the following:

\[
\begin{align*}
\lim_{x_1 \to -\infty} f^t_{I_1,t} &= 0, \\
\lim_{x_1 \to -\infty} f^t_{I_1,t} &= 1, \\
\lim_{x_2 \to \infty} f^t_{I_1,t} &= 0, \\
\lim_{\nu_A \to 1} \nu_A f^t_{I_1,t} &= 0, \\
\frac{\partial f^t_{I_1,t}}{\partial \nu_A} \bigg|_{\nu_A=0} &= 0.
\end{align*}
\]

Finally, when \( x_2 \to -\infty \), then the second dividend tree becomes irrelevant and \( f^t_{I_1,t} \) converges to the case of Bhamra (2007). The other boundary conditions are justified as follows:

1. \( \lim_{x_1 \to -\infty} f^t_{I_1,t} = 0 \) and \( \lim_{x_2 \to \infty} f^t_{I_1,t} = 0 \): When \( x_2 \to \infty \), I must be that \( x_1 \to -\infty \). When \( x_1 \to -\infty \), the first dividend stream becomes irrelevant so investors aren’t willing to pay anything to own the stock.

2. \( \lim_{x_1 \to -\infty} f^t_{I_1,t} = \frac{1}{s} \): In this case there is a single dividend tree and complete markets (the
constraint becomes irrelevant), so

\[
S_1 = \frac{D_1}{\delta} = D_M (\nu_A f_{1,t}^A + \nu_I f_{1,t}^T)
\]

\[
\Rightarrow \frac{1}{\delta} = \nu_{A,t} \left( \frac{1}{\delta} \right) + (1 - \nu_{A,t}) f_{1,t}^T = f_{1,t}^T.
\]

**3.** \( \lim_{\nu_A \to 1} \nu_{I,t} f_{1,t}^T = 0 \): When \( \nu_A = 1 \), agent \( A \), which faces no constraint, consumes all dividends so markets are complete. Therefore \( f_{1,t}|_{\nu_A=1} = f_{1,t}^A \) so this boundary condition must hold.

**4.** \( \frac{\partial f_{1,t}^T}{\partial \nu_A} \bigg|_{\nu_A=0} = 0 \): As \( \nu_A \to 0 \), indexers consume all dividends. However, they have a worst investment opportunity set than active investors, so this can’t hold for more than an instant. Therefore this boundary condition must be a reflecting boundary condition.

**A.10 Matching moments**

I now have expressions for both \( f_{i,t}^A \) and \( f_{i,t}^T \). I have a closed form expression for \( f_{i,t}^A \) that depends on exogenous parameters and state variables, which is easy to evaluate numerically. For \( f_{i,t}^T \), I have a PDE that can be approximated. However, the current form of that solution depends on the endogenously determined \( \bar{\sigma} \) because of the dependence on \( \bar{\sigma}_A \). I have that \( S_{i,t} = D_{M,t} f_{i,t} \), so

\[
dS_i = D_{M} df_i + f_i dD_M + df_i dD_M,
\]

\[
\frac{dS_i}{S_i} = \frac{df_i}{f_i} + \frac{dD_M}{D_M} + \frac{df_i}{f_i} \frac{dD_M}{D_M},
\]

where

\[
\frac{dD_M}{D_M} = \mu_D dt + \bar{\sigma}_D d\bar{Z}
\]

and \( \bar{\sigma}_D = s_1 \bar{\sigma}_{D_1} + s_2 \bar{\sigma}_{D_2} + (1-s_1-s_2) \bar{\sigma}_{D_3} \). I know that \( f_{i,t} \) is a function of exogenous parameters and state processes \( s_1 \), \( s_2 \) and \( \nu_A \), therefore from Itô’s Lemma I get:

\[
df_i = \left[ \mu_{\nu_A} \frac{\partial f_i}{\partial \nu_A} + \mu_{s_1} \frac{\partial f_i}{\partial s_1} + \mu_{s_2} \frac{\partial f_i}{\partial s_2} + \frac{1}{2} \left( \sigma_{\nu_A}^2 \frac{\partial^2 f_i}{\partial \nu_A^2} + \sigma_{s_1}^2 \frac{\partial^2 f_i}{\partial s_1^2} + \sigma_{s_2}^2 \frac{\partial^2 f_i}{\partial s_2^2} \right) + 2 \sigma_{\nu_A} \bar{\sigma}_{s_1} \frac{\partial f_i}{\partial \nu_A \partial s_1} + 2 \sigma_{\nu_A} \bar{\sigma}_{s_2} \frac{\partial f_i}{\partial \nu_A \partial s_2} + 2 \bar{\sigma}_{s_1} \bar{\sigma}_{s_2} \frac{\partial^2 f_i}{\partial s_1 \partial s_2} \right) dt
\]

\[
+ \left[ \sigma_{\nu_A} \frac{\partial f_i}{\partial \nu_A} + \bar{\sigma}_{s_1} \frac{\partial f_i}{\partial s_1} + \bar{\sigma}_{s_2} \frac{\partial f_i}{\partial s_2} \right] d\bar{Z}.
\]
From the definition of stock return process, I also have that
\[
\frac{dS_i}{S_i} = \left[ \mu_i - D_i \right] dt + \sigma_i dZ
\]  
(117)

where
\[
\mu_i = \frac{1}{f_i} \left[ \mu_{\nu,1} \frac{\partial f_i}{\partial \nu} + \mu_s \frac{\partial f_i}{\partial s} + \mu_{\nu,2} \frac{\partial f_i}{\partial \nu} + \frac{1}{2} \left( \sigma_\nu \frac{\partial^2 f_i}{\partial \nu^2} + \sigma_s \frac{\partial^2 f_i}{\partial s^2} + \sigma_{\nu,1} \frac{\partial^2 f_i}{\partial \nu \partial s} + \sigma_{\nu,2} \frac{\partial^2 f_i}{\partial \nu \partial s} \right) \right] + \mu_D,
\]  
\[
\sigma_i = \frac{1}{f_i} \left[ \sigma_{\nu,1} \frac{\partial f_i}{\partial \nu} + \sigma_s \frac{\partial f_i}{\partial s} + \sigma_{\nu,2} \frac{\partial f_i}{\partial s} \right] + \sigma_D.
\]  
(118)

Note that the expression I have for \( \sigma_\lambda \) from (61) is a function of both \( \sigma \) and equilibrium price ratio \( f_1/f_2 \), since \( \omega_i^1 = 1 + f_1/f_2 \) and \( \omega_i^2 = 1 + f_2/f_1 \). I first use the definitions of \( \sigma \) and \( \sigma_\lambda \) to create perturbation expansions of these moments as a function of \( f_1, f_2, f_3 \) and their own expansions. Substituting these expansions in the PDE (109), I create a perturbation expansion of the PDE, and then solve by equating terms in the different power of \( \epsilon \). The result is the closed-form approximation
\[
f_i^T = f_i^A + \frac{1}{2 (s_1 + s_2)} \nu A \sigma \left( 2 (-1 + s_1 + s_2) \left( s_2 + 2 \left( s_1^2 + s_1 (-1 + s_2) + s_2^2 \right) \right) + (s_1 + s_2) \left( 1 + 2 s_1^2 + 2 (-1 + s_2) s_2 + s_1 (-3 + 2 s_2) \right) \nu A \right) (1 - \rho_D) \sigma_D^2 + O(\epsilon^4).
\]  
(120)

As in the unconstrained economy, I find \( f_2^T \) by symmetry and \( f_3^A \) by \( f_3^A = \frac{1}{2} - f_1^T - f_2^T \). A drawback of the use of a perturbation expansion is that it is impossible to guarantee that the boundary conditions will be satisfied. It is easy to see that in this case (114) is not satisfied, which means that the approximation will not be valid in the neighbourhood of \( \nu_A = 0 \). Since this region is not economically important for the current analysis, this does not pose a problem as long as the analysis focuses on values of \( \nu_A \) that are away from that boundary.

### A.11 Vector notation

This section introduces the two different vector bases I use in the proofs. While not a necessary read, this section is a useful appendix for understanding the proofs. The reason for using...
different bases is to simplify certain steps of the proof. Steps involving stock returns are easier to solve under the market basis. However, when solving for equilibrium stock return dynamics, the dividend basis is more appropriate. The dividend processes in (1) can be represented as a vector

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D \text{1'}dZ_{D,t},$$

(121)

where \( \frac{dD_t}{D_t} \) is a vector with \( \frac{dD_{i,t}}{D_t} \) as the \( i \)-th element and \( dZ_{D,t} \) is a vector with \( dZ_{D_{i,t}} \) as the \( i \)-th element. Since the \( dZ_{D_{i,t}} \) can be correlated, we can represent the correlation matrix of \( dZ_{D,t} \) as

$$C_{D_t} = \begin{bmatrix} 1 & \rho_D & \rho_D \\ \rho_D & 1 & \rho_D \\ \rho_D & \rho_D & 1 \end{bmatrix}.$$

Stock returns in (4) can also be represented in vector notation

$$dR_t = \mu_t dt + \sigma_t dZ_t,$$

where \( dR_t, \mu_t \) and \( dZ_t \) are vectors with \( dR_{i,t}, \mu_{i,t} \) and \( dZ_{i,t} \) as the \( i \)-th element and \( \sigma_t \) is a diagonal matrix with \( \sigma_{i,t} \) as the \( i \)-th diagonal element. The \( dZ_t \) BMs are correlated with correlation matrix

$$C_t = \begin{bmatrix} 1 & \rho_{t,12} & \rho_{t,13} \\ \rho_{t,12} & 1 & \rho_{t,23} \\ \rho_{t,13} & \rho_{t,23} & 1 \end{bmatrix}.$$

### A.11.1 Rotation matrix

It is often easier to deal with independent Brownian motions (BM) than correlated ones. It is possible to transform a multivariate BM to a vector of independent BM using a rotation matrix. Under that transformation, drifts, variances and covariances of Itô processes are invariant. Consider the three-dimensional multivariate BM \( Z = [Z_1 \ Z_2 \ Z_3]' \) with correlation matrix

$$C = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

Using the Cholesky decomposition, we can construct a rotation matrix \( K \) to transform \( Z \) into a three-dimensional vector of independent BM. From the Cholesky decomposition, we get the lower triangular matrix \( L \) such that \( LL' = C \). The matrix \( L \) is often used to generate correlated BM from independent ones such that \( Z = LX \). In this case, I am interested in the inverse process, \( X = KZ \) where \( K = L^{-1} \).
Applying the Cholesky decomposition to the matrix $C$,

$$K = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{\rho_{12}}{\sqrt{1 - \rho_{12}^2}} & 0 & 0 \\
\frac{1}{\rho_{13} - \rho_{12} \rho_{23}} \left( \frac{-1 + \rho_{12}^2 + \rho_{13}^2}{\sqrt{1 - \rho_{12}^2}} \right) & \frac{1}{\rho_{12} \rho_{23}} \left( \frac{-1 + \rho_{12}^2 + \rho_{23}^2}{\sqrt{1 - \rho_{12}^2}} \right) & \frac{1}{1 - \rho_{12}^2} \left( \frac{1}{\sqrt{1 + \rho_{12}^2}} \right) \\
\end{bmatrix}.$$  \hspace{1cm} (122)

Changing the set of BMs using a rotation matrix is called a change of basis. Drift terms, total variances and covariances between processes are invariant under a change of basis. Note that if the initial BMs are uncorrelated (correlation terms in $C$ all equal to 0), then the rotation matrices $L$ and $K$ collapse to the identity matrix.

### A.11.2 Dividend basis

The BMs driving the dividend processes described in (121) are correlated. Consider $L_{D_t}$, the lower triangular matrix from the Cholesky decomposition of $C_{D_t}$, and it’s inverse $K_{D_t}$. Then I can rewrite (121) as

$$\frac{dD_t}{D_t} = \mu_{D} dt + \sigma_{D} dZ_{D_t}$$

$$= \mu_{D} dt + \sigma_{D} L_{D} Z_{D_t}$$

$$= \mu_{D} dt + \sigma_{D} Z_{D_t}$$

where $\sigma_{D} = \sigma_{D} L_{D}$ and $Z_{D_t} = K_{D} Z_{D_t}$. This transformation yields a new basis that I call the dividend basis. The variance matrix under the dividend basis can be written as

$$\sigma_{D} = \begin{bmatrix}
1 & 0 & 0 \\
\rho_{D} \sqrt{1 - \rho_{D}^2} & 0 \\
\rho_{D} \sqrt{1 - \rho_{D}^2} & \sqrt{3 - 2 \rho_{D} - \frac{2}{1 + \rho_{D}}} \\
\end{bmatrix}.$$  \hspace{1cm} (123)

### A.11.3 Market basis

Similarly, the BMs driving the market return processes in (11) might be correlated as they are determined endogenously. Consider $L_t$, the lower triangular matrix from the Cholesky
decomposition of $C_t$, and it's inverse $K_t$. Then I can write

$$dR_t = \mu_t dt + \sigma_t dZ_t$$
$$= \mu_t dt + \sigma_t L_t dZ_t$$
$$= \mu_t dt + \sigma_t dZ_t,$$

where $\sigma_t = \sigma_t L_t$ and $Z_t = K_t Z_t$. This transformation yields a new basis that I call the market basis. Under this basis,

\[
\begin{pmatrix}
\sigma_1 \\
\rho_{12} \sigma_2 \\
\rho_{13} \sigma_3 \\
\end{pmatrix} = \begin{pmatrix}
\sigma_1 \\
\rho_{12} \rho_{13} \sigma_3 \\
\rho_{13} \sigma_3 \\
\end{pmatrix}
\begin{pmatrix}
\sqrt{1 - \rho_{12}^2} \\
\rho_{12} \rho_{13} + \rho_{23} \rho_{12} \rho_{13} \ \
\rho_{13} \\
\end{pmatrix}
\begin{pmatrix}
\sigma_2 \\
\sigma_3 \\
\sigma_3 \\
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
\sqrt{1 - \rho_{12}^2} \\
\rho_{12} \rho_{13} + \rho_{23} \rho_{12} \rho_{13} \\
\rho_{13} \\
\end{pmatrix}
\begin{pmatrix}
\frac{\sigma_2}{\sqrt{1 - \rho_{12}^2}} \\
\frac{\sigma_3}{\sqrt{1 - \rho_{12}^2}} \\
\sigma_3 \\
\end{pmatrix}
\]

Note that the return process can also be written under the dividend basis as

$$dR_t = \mu_t dt + \sigma_t dZ_{D_t}$$

where $\sigma_t dZ_{D_t} = \sigma_t dZ_t = \sigma_t dZ_t$. $\sigma_t$ has the generic form:

$$\sigma_t = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} \\
\end{pmatrix}.$$

However, this leaves 9 unknowns to solve for in $\sigma_t$ (it is a $3 \times 3$ matrix), whereas the known structure of $\sigma_t$ leaves only 6 unknowns to solve for, namely $\sigma_1$, $\sigma_2$, $\sigma_3$, $\rho_{12}$, $\rho_{13}$ and $\rho_{23}$. 

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