The Role of Risk Aversion and Intertemporal Substitution in Dynamic Consumption-Portfolio Choice with Recursive Utility*

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Abstract

Our objective is to understand how risk aversion and elasticity of intertemporal substitution under recursive utility affect dynamic consumption-portfolio decisions. For a three-date model with a stochastic interest rate, we obtain an analytic solution for the optimal policies. We find that, in general, consumption and portfolio decisions depend on both risk aversion and elasticity of intertemporal substitution. The size of risk aversion relative to unity determines the sign of the intertemporal hedging portfolio, while elasticity of intertemporal substitution affects only its magnitude. The portfolio weight is independent of elasticity of intertemporal substitution only for the case of a constant investment opportunity set.

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1. Introduction

Recursive utility functions (Kreps and Porteus, 1978, Epstein and Zin, 1989), in contrast to expected utility functions, enable one to separate cleanly an investor’s risk aversion and elasticity of intertemporal substitution. The objective of this paper is to understand the role of these two distinct parameters for consumption and portfolio choice. Based on the analysis in a setting with a non-stochastic investment opportunity set (that is, constant interest rate, expected returns on risky assets, and volatilities of risky asset returns), Svensson (1989, p. 315) concludes that:

\[\text{Hence, the optimal portfolio depends on the risk aversion parameter but not on the intertemporal elasticity of substitution.}\]

In the real world, the investment opportunity set is not constant: interest rates, expected returns on risky assets, and volatilities of stock returns are stochastic. Hence, we wish to understand whether Svensson’s result is true also in a model with a stochastic investment opportunity set. In particular, the question we address in this paper is: how exactly do optimal consumption and portfolio decisions depend on risk aversion and elasticity of intertemporal substitution.

Our work is motivated by recent research in the areas of finance, for instance Weil (1989) and Campbell and Viceira (2002), and macroeconomics, for example Weil (1990), Obstfeld (1994) and Dumas and Uppal (2001), where the preferences of agents are characterized by recursive utility functions. With this utility function, analytic solutions for the dynamic consumption-portfolio problem are difficult to obtain when the investment opportunity set is stochastic.\(^1\) For instance, Svensson (1989) and Obstfeld (1994) obtain an explicit solution to the Bellman equation assuming the investment opportunity set is constant.

Weil (1990) considers an economy with a stochastic interest rate but in order to get closed-form results he assumes that the interest rate is identically and independently distributed over time; consequently, just as in the case of a constant investment opportunity set, the optimal portfolio is myopic and does not include a component to hedge against intertemporal changes in the investment opportunity set. Campbell and Viceira (2002) and Chacko and Viceira (2005) allow for a stochastic investment opportunity set but get analytic expressions that are only approximations to the true

\(^1\)Even with expected utility, a closed-form solution for the optimal portfolio weight in the presence of a stochastic investment opportunity set is available only under particular assumptions for the stochastic processes driving the state variables; see Liu (2005).
consumption and portfolio policies, while Dumas and Uppal (2001) solve their model numerically. Because these papers do not have an exact analytic solution, one cannot deduce from them the properties of the optimal dynamic consumption and portfolio policies. In particular, from these papers one cannot understand the economic forces driving the optimal consumption and portfolio decisions.

Our contribution is to derive the exact closed-form solutions for a stylized dynamic consumption and portfolio choice problem, which then allows us to identify precisely how different aspects of an investor’s preferences determine the optimal policies. We consider a three-date discrete-time setting where one can obtain an explicit solution even when the investment opportunity set is stochastic; this setting is similar to the one considered by Ingersoll (1987), but instead of using expected utility we use recursive utility.

The first finding of this paper is to show that, in general, the consumption and portfolio decisions depend on both risk aversion and the elasticity of intertemporal substitution. Thus, the result in Svensson (1989) does not generalize to the case where the investment opportunity set is stochastic; only in the special case where the investment opportunity set is constant (that is, returns are identically and independently distributed over time), is the optimal portfolio weight independent of the elasticity of intertemporal substitution, though even in this case the consumption decision depends on both risk aversion and elasticity of intertemporal substitution. Second, we find that the sign of the intertemporal hedging component in the optimal portfolio depends on the size of risk aversion relative to unity, while only the magnitude of the hedging portfolio depends on the elasticity of intertemporal substitution. Finally, we find that the effect of a change in the investment opportunity set on consumption depends on the income and substitution effects, and that the sign of the net effect depends on the size of the elasticity of intertemporal substitution relative to unity, while only the magnitude is affected by risk aversion; this result confirms the finding of the literature on consumption choice in non-stochastic settings and the literature on portfolio choice with a non-stochastic investment opportunity set.

For a stochastic investment opportunity set, Schroder and Skiadas (1999) under the assumption of complete markets, and Schroder and Skiadas (2003) for the case with portfolio constraints, characterize the solution to the continuous time consumption-portfolio problem in terms of the solution to a quasilinear partial differential equation. However, they solve this partial differential equation only for (i) the case of expected utility (that is, where risk aversion and elasticity of intertemporal substitution are given by the same parameter), and (ii) the case where the elasticity of intertemporal substitution is unity. Thus, they do not provide an explicit solution for general values of relative risk aversion and elasticity of intertemporal substitution. Giovannini and Weil (1989) examine the implications of recursive utility for equilibrium in the capital asset pricing model, while Ma (1993) shows the existence of equilibrium in an economy with multiple agents who have recursive utility; but neither paper provides an explicit expression for the optimal consumption and portfolio choice policies when the investment opportunity set is stochastic.
The rest of the paper is organized as follows. The model with a recursive utility function is described in Section 2. The consumption and portfolio problem of the investor is described in the first part of Section 3, with the rest of this section containing the solution to the problem when the investment opportunity set is constant and when it is stochastic. We conclude in Section 4. Detailed proofs for all propositions are presented in the appendix.

2. The model

In this section, we first define recursive utility functions. Following this, we describe the financial assets available to the investor, and the dynamic budget constraint that the investor faces when making her consumption and portfolio decisions.

2.1. Preferences

We assume that the preferences of the agent are recursive and of the form described by Epstein and Zin (1989). Hence the agent’s utility at time $t$, $U_t$, is given by

$$U_t = f(c_t, \mu_t (U_{t+1}))$$

and at the terminal date, $T$, by

$$U_T = B(W_T),$$

where $f$ is an aggregator function, $\mu_t (U_{t+1})$ is the certainty equivalent of the distribution of time $t+1$ utility, $U_{t+1}$, conditional upon time-$t$ information, and $B(W_T)$ is the bequest function.

We choose the aggregator

$$f(c, v) = \begin{cases} \left[(1 - \beta)c^\rho + \beta v^\rho\right]^{1/\rho}, & \rho \leq 1; \rho \neq 0, \\ (1 - \beta) \ln c + \beta \ln v, & \rho = 0, \end{cases}$$

where $\beta > 0$, and we specify the certainty equivalent of a random variable $x$ to be

$$\mu_t(x) = \begin{cases} (E_t x^\alpha)^{1/\alpha}, & \alpha \leq 1; \alpha \neq 0, \\ \exp(E_t \ln x), & \alpha = 0. \end{cases}$$

We also choose the bequest function

$$B(W_T) = f(W_T, 0), \quad \rho \neq 0.$$
Therefore, from (3) and (4), for the general values of $\rho$ and $\alpha$ we have that

$$U_t = \left[ (1 - \beta) C_t^\rho + \beta \left( E_t U_{t+1}^\alpha \right)^{\rho/\alpha} \right]^{1/\rho}, \quad \rho \leq 1; \rho \neq 0, \alpha \leq 1; \alpha \neq 0,$$

and for the cases where $\alpha$ and/or $\rho$ take the value of zero:

$$U_t = \begin{cases} 
(1 - \beta) C_t^\rho + \beta \left( \exp E_t [\ln U_{t+1}] \right)^{\rho/\alpha}, & \rho \leq 1; \rho \neq 0, \alpha = 0; \\
(1 - \beta) \ln C_t + \beta \ln \left( \left( E_t U_{t+1}^\alpha \right)^{1/\alpha} \right), & \rho = 0, \alpha \leq 1; \alpha \neq 0; \\
(1 - \beta) \ln C_t + \beta E_t \ln U_{t+1}, & \rho = 0, \alpha = 0.
\end{cases}$$

Note that according to the above specification, the relative risk aversion of the agent is given by $1 - \alpha$ and the elasticity of intertemporal substitution by $1/(1 - \rho)$. Hence, the recursive formulation allows one to disentangle the effects of relative risk aversion and the elasticity of intertemporal substitution.  

3 On the other hand, when $\rho = \alpha \neq 0$, equation (6) reduces to the power specification of expected utility

$$U_t = \left[ (1 - \beta) E_t \left( \sum_{j=0}^{T-1} \beta^j C_{t+j}^\alpha + \beta^{T-t} W_T^\alpha \right) \right]^{1/\alpha},$$

where relative risk aversion is $1 - \alpha$ and the elasticity of intertemporal substitution is $1/(1 - \alpha)$, so that both are determined by the same parameter, $\alpha$.

2.2. Financial Assets

Let there be a riskless asset with return $R$, and denote the time-$t$ price of the riskless security by $P_{0,t}$. In addition to the riskless asset, there are $n$ risky assets, whose rates of return are given by

$$\frac{P_{i,t+1}}{P_{i,t}} = z_{i,t+1}, \quad i \in \{1, \ldots, n\},$$

where $z_{i,t+1}$ is a random variable and $P_t$ is the price of the $i$’th risky asset. Note that the price at $t + 1$ is cum dividend.

3 Observe that by defining $V_t = h(U_t)$, where $h$ is increasing monotonic, we obtain a set of preferences ordinally equivalent to those represented by $U_t$. Hence, $V_t = h\left( f(e^{-\mu_t} (h^{-1}(V_{t+1}))) \right)$. We choose $h$ such that $V_t = h(U_t) \equiv U_t^\alpha / \alpha$, so that the value function for $V_t$, given by $\tilde{J}_t = h(J_t)$ corresponds with our usual notion of indirect utility with its standard interpretations: for example, the expression $-\alpha \tilde{J}_W / \tilde{J}_W$ gives relative risk aversion, $1 - \alpha$. See also Epstein and Zin (1989, p. 948) for a similar discussion.
2.3. Evolution of wealth and the budget constraint

An investor with wealth $W_t$ chooses to consume $C_t$ and buy $N_{i,t}$ shares of each asset, subject to the constraint

$$W_t - C_t \equiv I_t = \sum_{i=0}^{n} N_{i,t} P_{i,t}. \quad (10)$$

Thus, the proportion of wealth (net of consumption) invested in the $i$'th asset is given by $w_{i,t} = N_{i,t} P_{i,t}/I_t$. Then, the dynamic budget constraint can be written as:

$$W_{t+1} = I_t \sum_{i=0}^{n} w_{i,t} z_{i,t+1}$$

$$= [W_t - C_t] \left[ w_{0,t} R + \sum_{i=1}^{n} w_{i,t} z_{i,t+1} \right]$$

$$= [W_t - C_t] \left[ \left( 1 - \sum_{i=1}^{n} w_{i,t} \right) R + \sum_{i=1}^{n} w_{i,t} z_{i,t+1} \right]$$

$$= [W_t - C_t] \left[ \sum_{i=1}^{n} w_{i,t} (z_{i,t+1} - R) + R \right], \quad (11)$$

where the constraint $\sum_{i=0}^{n} w_{i,t} = 1$ has been substituted out.

3. The intertemporal consumption and portfolio choice problem

The objective of the agent is to maximize her lifetime expected utility by choosing consumption, $C_t$, and the proportions of her wealth to invest in the $n$ risky assets, $w_{i,t}$, subject to the budget constraint given in equation (11). In Section 3.1, we describe a general model and characterize the value function and optimal consumption and portfolio policies in this model. We then specialize this general setup: in Section 3.2, we consider the case where the investment opportunity set is constant, and in Section 3.3, we consider the case where the investment opportunity set is stochastic. For both cases, we derive the optimal consumption and portfolio policies and study how they depend on risk aversion and elasticity of intertemporal substitution.
3.1. Characterization of the solution to the general problem

We define the optimal value of utility in (6) as a function $J$ of current wealth, $W_t$ and time, $t$. If $\alpha \neq 0$ and $\rho \neq 0$, the Bellman equation takes the form

$$J_t(W_t) = \sup_{C_t, w_t} \left[ (1 - \beta) C_t^\rho + \beta \left( E_t J_{t+1}^\alpha (W_{t+1}) \right)^{\rho/\alpha} \right]^{1/\rho}, \quad (12)$$

where $w_t = \{w_{0,t}, \ldots, w_{n,t}\}$ is the vector of portfolio weights.

The proposition below describes the functional form taken by the value function and also the first-order conditions for the optimal consumption and portfolio policies.

**Proposition 1** The value function is given by

$$J_t(W_t) = (1 - \beta)^{\frac{1}{\rho}} a_t^{\frac{\rho-1}{\rho}} W_t, \quad (13)$$

where

$$a_t = \left( 1 + \left[ \beta \left( E_t Z_{t+1}^\alpha \left( a_{t+1}^{\frac{\rho-1}{\rho}} \right)^{\frac{1}{1-\rho}} \right) \right]^{-1} \right), \quad a_T = 1, \quad (14)$$

and the return on the portfolio is

$$Z_{t+1} = \sum_{i=0}^{n} w_{i,t} z_{i,t+1}. \quad (15)$$

The optimal consumption policy is given by

$$C_t = a_t W_t, \quad (16)$$

and the optimal portfolio policy is defined by the condition

$$E_t \left[ a_{t+1}^{\frac{\alpha(\rho-1)}{\rho}} Z_{t+1}^{\alpha-1} (z_{i,t} - R) \right] = 0. \quad (17)$$

The results in this proposition are similar to results in Epstein and Zin (1989) who consider an infinite-horizon setting, and so the proposition above shows that the first-order conditions for consumption and portfolio choice derived in Epstein and Zin hold also for a finite time horizon.
3.2. Solution for a constant investment opportunity set

We consider first the case where the investment opportunity set is constant. This allows us to identify explicitly the value function and the optimal portfolio and consumption choices. We assume that there is only one risky asset with two equally likely payoffs, \( h > R \) and \( k < R \). In this case, we have the following.\(^4\)

**Proposition 2** The value function when there is only a single risky asset with two equally likely payoffs, \( h > R \) and \( k < R \) is given by

\[
J_t(W_t) = (1 - \beta)^{\frac{1}{\alpha}} a_t^{\frac{\rho - 1}{\rho}} W_t, \quad t \in \{1, \ldots, T\},
\]

where

\[
a_t = \frac{1 - (\beta \text{CEQ}^\rho)^{\frac{1}{\rho}}}{1 - (\beta \text{CEQ}^\rho)^{(T-t+1)/(1-\rho)}},
\]

\[
\text{CEQ} = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} R(h-k) \left[ (h-R)^{-\frac{1}{1-\alpha}} + (R-k)^{-\frac{1}{1-\alpha}} \right]^{\frac{1-\alpha}{\alpha}}.
\]

The optimal consumption policy is given by

\[
C_t = a_t W_t,
\]

and the optimal portfolio policy by

\[
w_{1,t} = \frac{R(h-k)^{-\frac{1}{1-\alpha}} - (h-R)^{-\frac{1}{1-\alpha}}}{(R-k)^{-\frac{1}{1-\alpha}} + (h-R)^{-\frac{1}{1-\alpha}}}.
\]

From equation (21), we see that the investor consumes a proportion of her wealth, which from the definition of \( a_t \) in equation (19), depends on her rate of time preference, relative risk aversion and elasticity of intertemporal substitution, the investment horizon and characteristics of the risky and riskless assets (through \( \text{CEQ} \)). Thus, even in the case of a constant investment opportunity set, the consumption policy depends on both risk aversion and the elasticity of intertemporal substitution.

\(^4\)Note that the value function corresponding to \( U_t \) is given by \( J_t \), which is linear in wealth, just as in Epstein and Zin (1989). The value function corresponding to \( V_t \) (defined in Footnote 3) is \( \hat{J}_t \), which is given by \( \hat{J}_t = J_t^\rho / \rho \). Observe also that in Proposition 2, by setting \( \rho = \alpha \), we obtain the well known results for power utility and by taking appropriate limits, we obtain the results for logarithmic utility.
As is well known, the marginal propensity to consume, given by \( a_t \), depends on elasticity of intertemporal substitution because it is this parameter that determines the desire of the investor to smooth consumption over time. The reason why consumption depends also on risk aversion is because risk aversion affects the certainty-equivalent value of wealth (given by CEQ in equation (20)), which then influences how much the investor wishes to consume.

On the other hand, for the case of a constant investment opportunity set the optimal portfolio in equation (22) depends only on the characteristics of the risky and riskless assets and the investor’s relative risk aversion and not on the investor’s intertemporal elasticity of substitution, which is consistent with the results in Svensson (1989) and Schroder and Skiadas (1999, 2003).

The reason the investor’s elasticity of intertemporal substitution does not play a role in portfolio choice is the following. Portfolio decisions are made by maximizing the certainty equivalent of future utility (that is, by choosing a portfolio to smooth utility across states). Future utility depends on a function of the state multiplied by the portfolio return in that state. This function of the state is where elasticity of substitution appears. However, when the investment opportunity set is constant, the function of the state multiplying the portfolio return is the same across states, and hence, does not influence the optimization. Therefore, only relative risk aversion influences portfolio choice—the term containing the elasticity of intertemporal substitution parameter drops out. Hence, in the case of a constant investment opportunity set, maximizing the certainty equivalent of future utility is the same as maximizing the certainty equivalent of future returns. This maximized value is given in equation (20).

### 3.3. Solution for a stochastic investment opportunity set

We now examine the effects of a stochastic investment opportunity set on optimal portfolio and consumption decisions. We do this by extending to the case of recursive utility an example considered in Ingersoll (1987) for the case of expected utility. This is a simple three-date model with consumption and portfolio decisions at \( t = \{0, 1\} \) and bequest at \( t = 2 \).

Suppose that at \( t = 0 \) there are two assets available to the investor: a one-period riskfree asset with return \( R \) and a one-period risky asset. At \( t = 1 \), the return on the risky asset is either 0 or \( 2R \), with equal probability. If the return on the risky asset is 0, then the one-period riskfree rate of return (from \( t = 1 \) to \( t = 2 \)) is \( RU \); on the other hand, if the return on the risky asset is \( 2R \), then

\[ RU \]

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5This result is also shown in Weil (1989).
the riskfree rate of return (from \( t = 1 \) to \( t = 2 \)) is \( R_D < R_U \); thus, the riskless interest rate at \( t = 1 \) is negatively correlated to the return on the risky asset. It will be important to keep this in mind when interpreting the results because investing more in the risky asset leads to a portfolio that is less risky than the one that has a smaller quantity invested in the risky asset. The financial assets available at each date and their payoffs are illustrated in Figure 1.

The model is deliberately specialized so that demand for the risky asset would be zero if the investment opportunity set were constant: at \( t = 0 \), the expected return on the risky asset is the same as that on the riskless asset, and therefore, no single-period risk-averse investor would invest in the risky asset. However, multiperiod investors may hold the risky asset in order to hedge against changes in the investment opportunity set. Hence, the demand for the risky asset arises only from the desire for intertemporal hedging; we refer to this portfolio as the “hedging portfolio.”

We start by considering the effect of a change in the investment opportunity set, through a change in the riskless interest rate, on consumption. This effect can be decomposed into “substitution” and “income” effects.

**Proposition 3** The substitution effect is given by

\[
\frac{\partial C_1}{\partial R} \bigg|_{J_1} = -a_1^2 W_1 \beta R^{1-\rho} \frac{R^{\rho-1}}{1 - \rho}
\]

(23)

where

\[
a_1 = \left[ 1 + (\beta R^\rho)^{1-\rho} \right]^{-1},
\]

(24)

and the income effect by

\[
(W_1 - C_1) R^{-1} \frac{\partial C_1}{\partial W_1} = W_1 (1 - a_1) R^{-1} a_1.
\]

(25)

The sum of these two effects is given by

\[
\frac{dC_1}{dR} = -a_1^2 W_1 R^{-1} (\beta R^\rho)^{1-\rho} \frac{\rho}{1 - \rho}.
\]

(26)

Note that the substitution effect is always negative, whereas the income effect is always positive. The intuition for this is well-known: as the riskfree rate increases, future consumption becomes cheaper relative to current consumption; hence current consumption decreases (substitution effect).
However, the increase in the riskfree rate increases overall wealth, which leads to an increase in current consumption (income effect). When $\rho > 0$ (elasticity of intertemporal substitution is greater than unity), the substitution effect is dominant and an increase in the riskfree rate leads to a decrease in current consumption. For the case where $\rho < 0$ (elasticity of intertemporal substitution is less than unity), the income effect is dominant and an increase in the riskfree rate, leads to an increase in current consumption rises. When $\rho = 0$, the income effect exactly offsets the substitution effect, which is a well-known result for the case of logarithmic expected utility preferences ($\rho = \alpha = 0$) where both risk aversion and elasticity of substitution equal unity. The example shows that the result for logarithmic expected utility depends only on having elasticity of intertemporal substitution equal to unity, and is independent of whether the relative risk aversion of the investor is smaller or greater than unity. These results are similar to those in Weil (1990).  

We now derive the expression for the optimal portfolio weight.

**Proposition 4** The intertemporal hedging demand for the risky asset (Asset 1) at time 0 is given by

$$w_{1,0} = \frac{1 + (\beta R_D^\rho)^{\frac{1}{\rho - 1}}} {1 + (\beta R_D^\rho)^{\frac{1}{\rho - 1}}} \frac{1 + (\beta R_U^\rho)^{\frac{1}{\rho - 1}}} {1 + (\beta R_U^\rho)^{\frac{1}{\rho - 1}}}$$

(27)

For the special case where the investor derives no utility from intermediate consumption and cares only about consumption on the terminal date ($t = 2$), the expression for the optimal intertemporal hedging demand, $w_{1,0}^N$, is

$$w_{1,0}^N = \frac{R_D^{\frac{\alpha}{\rho - 1}} - R_U^{\frac{\alpha}{\rho - 1}}} {R_D^{\frac{\alpha}{\rho - 1}} + R_U^{\frac{\alpha}{\rho - 1}}}.$$  

(28)

Observe that in the general case, the optimal portfolio weight, which is given in equation (27), depends on both the parameter controlling relative risk aversion, $\alpha$ and the parameter driving elasticity of intertemporal substitution, $\rho$. We now explain the role of these two parameters. We first explain this in terms of how recursive utility is defined, and then provide some economic intuition.

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6Observe that both the substitution and income effects of a change in the riskfree rate, $R$, on consumption are independent of risk aversion. The reason for this is that in the specialized model we are considering in this section of the paper, all uncertainty is resolved at $t = 1$.  

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From the definition of the aggregator in equation (3), we see that it is the parameter for elasticity of intertemporal substitution, $\rho$, that determines the desire for smoothing consumption over time. And, the definition of the certainty equivalent in equation (4) shows that it is the parameter for relative risk aversion, $\alpha$, that determines the desire to smooth future utility across states. Portfolio decisions are made to maximize the certainty equivalent of future utility (that is, by smoothing utility across states), and thus, $\alpha$ plays a role in determining the optimal portfolio. Future utility depends on a function of the state ($R_U$ or $R_D$) multiplied by the portfolio return in that state. This function of the state is where elasticity of substitution appears. When the investment opportunity set is stochastic, the function of the state multiplying the portfolio return influences the optimization. Hence, elasticity of substitution affects portfolio choice through its affect on future utility.\(^7\)

Note from equation (28) that when there is no intermediate consumption and the investor cares only about utility from terminal consumption, even then the investor will choose to hold a hedging portfolio. This is because the objective of the investor is not just to smooth consumption but rather to smooth utility across states. However, as is evident from equation (28), in this case the hedging portfolio is independent of the parameter $\rho$.

Next, we study the sign of the hedging portfolio.

**Proposition 5** The intertemporal hedging portfolio demand for the risky asset at $t = 0$, given by $w_{1,0}$ in equation (27), and $w_{1,0}^N$ in equation (28) for the case without intermediate consumption, are strictly negative if and only if relative risk aversion is strictly smaller than unity; strictly positive if and only if relative risk aversion is strictly larger than unity; and, zero if and only if relative risk aversion is unity.

We now provide some economic intuition for why the sign of the hedging demand for the risky asset depends only on the investor’s relative risk aversion relative to unity.\(^8\) The hedging behavior that we see stems from three facts: (i) because investors are risk averse, they wish to smooth utility across states; (ii) utility is affected by a change in the investment opportunity set, which in our case corresponds to a change in the riskless interest rate; and, (iii) in the model specified here, the

\(^7\)See also Liu (2005) for a discussion of this, albeit in a setting with expected utility.

\(^8\) Table A6 in the appendix to Campbell and Viceira (2001) gives numerical values for the hedging portfolio for different values of relative risk aversion and elasticity of intertemporal substitution; this table confirms that the sign of the hedging portfolio depends on whether relative risk aversion is larger or smaller than unity.
return on the risky asset and the change in the interest rate are negatively correlated, so a portfolio with a positive position in the risky asset is less risky than a portfolio with a negative position in the risky asset.

We start by discussing the case where $\alpha = 0$ (logarithmic risk aversion). In this case, maximizing the certainty equivalent of future utility is equivalent to maximizing the expected value of the logarithm of future utility (see equation (4)). Future utility depends on a function of the state ($R_U$ or $R_D$) multiplied by the portfolio return in that state; but, a logarithmic certainty equivalent, turns this product into a sum. Hence, the portfolio decision becomes independent of the state of the economy, and thus, no hedging is required:  

$$w_{1,0} = 0,$$  \hspace{1cm} (29)

As risk aversion increases beyond unity ($\alpha < 0$), investors wish to hold a less risky portfolio. They do so by holding a positive quantity of the risky asset. Because the correlation of the return on the risky asset and the interest rate prevailing at $t = 1$ is negative, holding a positive amount of the risky asset leads to a portfolio that has less overall risk.

Investors who are less risk averse than the investor with logarithmic risk aversion ($\alpha > 0$), are willing to hold a more risky portfolio. They do so by taking a negative position in the risky asset. Because the correlation of the return on the risky asset and the interest rate prevailing at $t = 1$ is negative, holding a short position in the risky asset leads to a portfolio that has more overall risk.

Note that while the elasticity of intertemporal substitution parameter, $\rho$, influences the magnitude of the hedging demand via its effect on future utility across states, it does not affect the sign of the hedging demand, which, as explained above, is determined only by risk aversion relative to unity. The reason why the sign of the hedging portfolio does not depend on whether elasticity of intertemporal substitution is greater or small than unity is because this parameter influences utility only over time but not across states. And, portfolio choice is driven only by the desire to smooth utility across states. So, while elasticity of intertemporal substitution determines whether it is the substitution or income effect that dominates, once the saving decision has been made, the sign of the portfolio decision is independent of the amount of savings, though the magnitude of the hedging portfolio is not. In fact, we can show that when elasticity of intertemporal substitution is

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9. This is a generalization of the well-known result for expected utility that when the utility function is logarithmic (so that both relative risk aversion and the elasticity of intertemporal substitution are unity), the hedging demand is zero.
unity, the hedging demand is non-zero; it is given by

\[ w_{1.0} = \frac{R_D^{1-\alpha}}{R_D^{1-\alpha} + R_U^{1-\alpha}} \]

and the sign of this expression depends on relative risk aversion, just as it did for the general case.

4. Conclusion

We study the role of risk aversion and intertemporal substitution in the optimal consumption and portfolio problem of an individual investor when the investment opportunity set is stochastic. Using a three-date model we find a solution in closed form and show that the consumption and portfolio decisions depend on both risk aversion and the elasticity of intertemporal substitution. Our main finding is that the size of risk aversion relative to unity determines the sign of the intertemporal hedging component in the optimal portfolio and the elasticity of intertemporal substitution affects only the magnitude of the hedging portfolio. While the model we have considered was specialized in order to obtain closed-form results, the intuition for the effects of risk aversion and intertemporal substitution that we identify is general and would be true also in other models of intertemporal portfolio choice.

\[^{10}\text{The derivation of this expression is provided in the last section of the appendix.}\]
A. Detailed proofs for all propositions

Proof of Proposition 1

The Bellman equation is

\[ J_t(W_t) = \sup_{C_t, W_t} f(C_t, \mu_t(J_{t+1}(W_{t+1}))) \]

\[ = \sup_{C_t, W_t} \left[ (1 - \beta) C_t^\rho + \beta \left( E_t J_{t+1}^\rho (W_{t+1}) \right)^{\rho/\alpha} \right]^{1/\rho}, \tag{A.1} \]

and the intertemporal budget constraint is

\[ W_{t+1} = I_t \left[ \sum_{i=1}^n w_{i,t} (z_{i,t+1} - R) + R \right] = I_t Z_{t+1}, \tag{A.2} \]

where \( I_t = W_t - C_t \) and \( Z_{t+1} = \sum_{i=1}^n w_{i,t} (z_{i,t+1} - R) + R. \)

From the homotheticity of utility we can write

\[ J_t(W_t) = A h_t W_t. \tag{A.3} \]

for some constant \( A \) and some function \( h_t \) independent of wealth. From the terminal condition

\[ J_T(W_T) = (1 - \beta)^{1/\rho} W_T \equiv B(W_T), \tag{A.4} \]

it follows that \( A = (1 - \beta)^{1/\rho} \), and \( h_T = 1. \)

Writing \( C_t = a_t W_t \), and using the intertemporal budget constraint, we can simplify the Bellman equation to obtain

\[ A h_t = \sup_{a_t, W_t} f \left( a_t, A(1 - a_t) \mu_t(h_{t+1} Z_{t+1}) \right), \tag{A.5} \]

or

\[ h_t = \sup_{a_t, W_t} \left[ a_t^\rho + \beta (1 - a_t)^\rho \left( E_t \left( h_{t+1}^\alpha Z_{t+1}^\alpha \right) \right)^{\rho/\alpha} \right]^{1/\rho}. \tag{A.6} \]

From this we can derive the first-order conditions for optimality. The first order condition for consumption is given by

\[ \frac{\partial}{\partial a_t} f \left( a_t, A(1 - a_t) \mu_t(h_{t+1} Z_{t+1}) \right) = 0. \tag{A.7} \]
We simplify this expression to obtain
\[ a_t^{\rho-1} = \beta(1 - a_t)^{\rho-1} (\mu_t (h_{t+1} Z_{t+1}))^{\rho}. \]  \hfill (A.8)

This first-order condition equates the marginal benefit from an increase in current consumption to the marginal cost of the resultant decrease in investment. From (A.6), we know that at the optimum
\[ h_t = a_t + \beta(1 - a_t)^{\rho} (\mu_t (h_{t+1} Z_{t+1}))^{\rho}. \]  \hfill (A.9)

Combining equations (A.8) and (A.9) to eliminate \((\mu_t (h_{t+1} Z_{t+1}))^{\rho}\) yields the optimal proportion of wealth to consume:
\[ a_t = h_t^{\frac{\rho}{\rho-1}}. \]  \hfill (A.10)

The optimal portfolio is determined by maximizing \([a_t^{\rho} + \beta(1 - a_t)^{\rho} (\mu_t (h_{t+1} Z_{t+1}))^{\rho}]^{1/\rho}\), with respect to \(w_t\), which for \(\rho \neq 0\) is equivalent to maximizing \(\mu_t (h_{t+1} Z_{t+1})\). Hence the first-order condition for optimal portfolio choice is
\[ \frac{\partial}{\partial w_{i,t}} \mu_t (h_{t+1} Z_{t+1}) = 0. \]  \hfill (A.11)

This can be simplified to obtain
\[ E_t [h_{t+1} Z_{t+1}^{\rho-1} (z_{i,t+1} - R)] = 0. \]  \hfill (A.12)

Using * to denote the optimal choice, we define the return on the optimal portfolio:
\[ Z_{t+1}^{*} = \sum_{i=1}^{n} w_{i,t}^{*} (z_{i,t+1} - R) + R. \]  \hfill (A.13)

Substituting the above expression for \(Z_{t+1}^{*}\) into the first-order condition for consumption given in (A.8) and simplifying gives
\[ a_t = \frac{[\beta \mu_t^{\rho} (h_{t+1} Z_{t+1}^{*})]^{\frac{1}{\rho}}}{1 + [\beta \mu_t^{\rho} (h_{t+1} Z_{t+1}^{*})]^{\frac{1}{\rho-1}}} = \frac{1}{1 + [\beta \mu_t^{\rho} (h_{t+1} Z_{t+1}^{*})]^{\frac{1}{\rho-1}}}. \]  \hfill (A.14)

Equation (A.10) implies that
\[ h_t = a_t^{\frac{\rho}{\rho-1}}. \]  \hfill (A.15)
Substituting (A.15) into equation (A.14) gives

\[
a_t = \left( \frac{1}{1 + \left[ \beta \mu_t \left( \frac{a_\rho^{-1}}{a_\rho^{-1} Z_{t+1}} \right)^{\frac{1}{1-\rho}} \right]} \right)^{-1}
\]

\[
a_t = \left( 1 + \left[ \beta \left( E_t Z_{t+1}^{\alpha a_\rho^{-1}} \right)^{\frac{\alpha}{1-\rho}} \right]^{\frac{1}{1-\rho}} \right)^{-1}, \quad a_T = 1. \tag{A.16}
\]

Using equation (A.10), we can also write the value function as in equation (13) and the first-order condition for optimal portfolio choice as in equation (17).

**Proof of Proposition 2**

Given that the investment opportunity set is assumed to be constant, \( a_{t+1} \) is known at time \( t \). Hence, from (A.16), it follows that

\[
a_t^{-1} = 1 + (\beta \text{CEQ})^{\frac{1}{1-\rho}} a_{t+1}^{-1}, \tag{A.17}
\]

where

\[
\text{CEQ} = \mu_t(Z_{t+1}). \tag{A.18}
\]

Note that because the opportunity set is constant, CEQ is independent of time. We can solve the difference equation (A.17) to obtain

\[
a_t = \frac{1 - (\beta \text{CEQ})^{\frac{1}{1-\rho}}}{1 - (\beta \text{CEQ})^{\frac{1}{1-\rho}}}, \tag{A.19}
\]

where we have used the terminal condition \( a_T = 1 \).

We can further simplify equation (A.18), by finding the maximum value of \( \text{CEQ} = \mu_t(Z_{t+1}) \), which is related to finding the optimal portfolio. To see this, we note that the optimal portfolio at date \( t \) is chosen to maximize the certainty equivalent \( \mu_t(h_{t+1} Z_{t+1}) \). We know from equation (A.10) that \( h_{t+1} = a_{t+1}^{\frac{\alpha-1}{\rho}} \). Hence,

\[
\mu_t(h_{t+1} Z_{t+1}) = \mu_t(a_{t+1}^{\frac{\alpha-1}{\rho}} Z_{t+1}). \tag{A.20}
\]

The optimal consumption-wealth ratio is deterministic, so it follows that

\[
\mu_t(h_{t+1} Z_{t+1}) = a_{t+1}^{\frac{\alpha-1}{\rho}} \mu_t(Z_{t+1}). \tag{A.21}
\]
Hence, the optimal portfolio at date $t$ is chosen by maximizing $\text{CEQ} = \mu_t(Z_{t+1})$. Given that there is only one risky asset with two equally likely payoffs,

$$\text{CEQ} = \mu_t(Z_{t+1})$$

$$= (E_t(Z_{t+1})^\alpha)^{1/\alpha}$$

$$= \left( \frac{1}{2} [R + w_{1,t} (h - R)]^\alpha + \frac{1}{2} [R + w_{1,t} (k - R)]^\alpha \right)^{1/\alpha}. \quad (A.22)$$

The first-order condition for maximizing the above expression with respect to $w_{1,t}$ is given by

$$[R + w_{1,t} (h - R)]^{\alpha - 1} (h - R) = [R + w_{1,t} (k - R)]^{\alpha - 1} (R - k). \quad (A.23)$$

This first-order condition tells us that at the optimum, the marginal benefit from holding more of the risky asset in the up-state is equal to that from holding more of the risky asset in the down-state.

Hence, the optimal portfolio weight for the risky asset is given by

$$w_{1,t} = R \frac{(R - k)^{-\frac{1}{\alpha}} - (h - R)^{-\frac{1}{\alpha}}}{(R - k)^{-\frac{\alpha}{1-\alpha}} + (h - R)^{-\frac{\alpha}{1-\alpha}}}. \quad (A.24)$$

Having obtained the optimal portfolio weight, $w_{1,t}$, we can now simplify the expression (A.22) for CEQ.

Simplifying the term $R + w_{1,t} (h - R)$ in the right-hand side of (A.22) yields

$$R + w_{1,t} (h - R) = R + R (h - R) \frac{(R - k)^{-\frac{1}{\alpha}} - (h - R)^{-\frac{1}{\alpha}}}{(R - k)^{-\frac{\alpha}{1-\alpha}} + (h - R)^{-\frac{\alpha}{1-\alpha}}}$$

$$= R \frac{(R - k)^{-\frac{\alpha}{1-\alpha}} + (h - R)^{-\frac{\alpha}{1-\alpha}} + (h - R) (R - k)^{-\frac{1}{\alpha}} - (h - R)^{-\frac{1}{\alpha}}}{(R - k)^{-\frac{\alpha}{1-\alpha}} + (h - R)^{-\frac{\alpha}{1-\alpha}}}$$

$$= R \frac{(R - k)^{-\frac{\alpha}{1-\alpha}} + (h - R) (R - k)^{-\frac{1}{\alpha}}}{(R - k)^{-\frac{1}{\alpha}} + (h - R)^{-\frac{1}{\alpha}}}$$

$$= R \frac{(R - k)^{-\frac{1}{\alpha}} (h - k)}{(R - k)^{-\frac{\alpha}{1-\alpha}} + (h - R)^{-\frac{\alpha}{1-\alpha}}}.$$
we obtain

\[
Q_t = \frac{(R - k)^{-\frac{\alpha}{1+\alpha}} + (h - R)^{-\frac{\alpha}{1+\alpha}} - (R - k)\left((k - R)(h - R)^{-\frac{1}{1+\alpha}}\right)}{(R - k)^{-\frac{\alpha}{1+\alpha}} + (h - R)^{-\frac{\alpha}{1+\alpha}}}
\]

Thus completes the proof.

Proof of Proposition 3

The standard two-good cross-substitution Slutsky equation, which measures the change in the quantity of the \(i\)'th good, \(Q_i\), with respect to a change in the price of the \(j\)'th good, \(P_j\), is

\[
dQ_i / dP_j = \partial Q_i / \partial P_j \bigg|_U - Q_j \partial Q_i / \partial W \bigg|_{P_i, P_j}.
\]  \tag{A.25}

The first term is the substitution effect and the second is the income effect. At \(t = 1\), \(Q_1 = C_1\), \(Q_2 = W_2\), \(P_1 = 1\), and \(P_2 = R^{-1}\). Substituting into the above expression and using \(dP_2/dR = -R^{-2}\) we obtain

\[
dC_1 / dR = \frac{dP_2}{dR} \frac{dC_1}{dP_2} = \frac{dP_2}{dR} \left[ \frac{\partial C_1}{\partial R} \bigg|_{P_1} \frac{\partial R}{\partial P_2} - W_2 \frac{\partial C_1}{\partial W_1} \bigg|_{R} \right]
\]

\[
= -R^{-2} \left[ -\frac{\partial C_1}{\partial R} \bigg|_{P_1} R^2 - W_2 \frac{\partial C_1}{\partial W_1} \bigg|_{R} \right]
\]

\[
= \frac{\partial C_1}{\partial R} \bigg|_{P_1} + W_2 R^{-2} \frac{\partial C_1}{\partial W_1} \bigg|_{R}
\]
\[
\frac{\partial C_1}{\partial R} \bigg|_{J_1} + (W_1 - C_1) R^{-1} \frac{\partial C_1}{\partial W_1} \bigg|_{R}.
\] (A.26)

We know that
\[
J_1(W_1) = (1 - \beta)\frac{1}{\rho} a_1^{\frac{\rho - 1}{\rho}} W_1 = (1 - \beta)\frac{1}{\rho} a_1^{\frac{\rho - 1}{\rho}} C_1 = (1 - \beta)\frac{1}{\rho} a_1^{\frac{1}{\rho}} C_1.
\] (A.27)

Hence
\[
\frac{\partial J_1(W_1)}{\partial C_1} = (1 - \beta)\frac{1}{\rho} a_1^{\frac{1}{\rho}}
\] (A.28)

and
\[
\frac{\partial J_1(W_1)}{\partial R} = -\frac{1}{\rho} (1 - \beta)\frac{1}{\rho} a_1^{\frac{1}{\rho} - 1} \frac{\partial a_1}{\partial R} C_1.
\] (A.29)

From the Implicit Function Theorem,
\[
\frac{\partial C_1}{\partial R} \bigg|_{J_1} = -\frac{\partial J_1(W_1)/\partial R}{\partial J_1(W_1)/\partial C_1}.
\] (A.30)

From equation (14), we know that
\[
a_t = \left\{ 1 + \left[ \beta \left( E_t(Z_t)^{\alpha} a_{t+1}^{\frac{2}{\rho} - 1} \right) \right] \right\}^{\frac{1}{1 - \rho}}, a_2 = 1,
\] (A.31)

because the date 2 is the final date. We also note that \(Z_1 = R\), because \(w_{1,1} = 0\). Hence
\[
a_1^{-1} = 1 + (\beta R^\rho)^\frac{1}{1 - \rho},
\] (A.32)

We now calculate
\[
\frac{\partial J_1(W_1)/\partial R}{\partial J_1(W_1)/\partial C_1} = -\frac{1}{\rho} (1 - \beta)\frac{1}{\rho} a_1^{\frac{1}{\rho} - 1} \frac{\partial a_1}{\partial R} C_1
\]
\[
= -\frac{1}{\rho} C_1 \frac{\partial a_1}{\partial R}
\]
\[
= \frac{C_1 \partial \log a_1^{-1}}{\rho \frac{\partial}{\partial R}}.
\] (A.33)
From equation (A.32), we obtain

\[
\frac{\partial \log a_{1}^{-1}}{\partial R} = \frac{1}{1 + (\beta R^\rho)^{1/\rho}} \frac{\partial}{\partial R} [ (\beta R^\rho)^{1/\rho} ]
\]

\[
= \frac{\rho}{1 - \rho} \frac{\beta^{1/\rho}}{1 + (\beta R^\rho)^{1/\rho}} R^{(\rho-1)\rho^{-1}}
\]

\[
= \frac{\rho}{1 - \rho} \beta^{1/\rho} a_1 R^{(\rho-1)\rho^{-1}}.
\]  

\[\text{(A.34)}\]

Hence

\[
\frac{\partial J_1(W_1) / \partial R}{\partial J_1(W_1) / \partial C_1} = \frac{a_1 C_1 \beta^{1/\rho} R^{(\rho-1)\rho^{-1}}}{1 - \rho}.
\]  

\[\text{(A.35)}\]

Therefore, we obtain that the substitution effect is:

\[
\left. \frac{\partial C_1}{\partial R} \right|_{J_1} = - \frac{\partial J_1(W_1) / \partial R}{\partial J_1(W_1) / \partial C_1} = - \frac{a_1^2 W_1 \beta^{1/\rho} R^{(\rho-1)\rho^{-1}}}{1 - \rho}.
\]  

\[\text{(A.36)}\]

The income effect is

\[
(W_1 - C_1) R^{-1} \frac{\partial C_1}{\partial W_1} = W_1 (1 - a_1) R^{-1} a_1.
\]  

\[\text{(A.37)}\]

Hence, the net effect is

\[
\frac{dC_1}{dR} = - \frac{a_1^2 W_1 \beta^{1/\rho} R^{(\rho-1)\rho^{-1}}}{1 - \rho} + W_1 (1 - a_1) R^{-1} a_1
\]

\[
= a_1^2 W_1 R^{-1} \left[ \frac{\beta^{1/\rho} R^{(\rho-1)\rho^{-1}}}{1 - \rho} + (\beta R^\rho)^{1/\rho} \right]
\]

\[
= -a_1^2 W_1 R^{-1} (\beta R^\rho)^{1/\rho} \left( \frac{1}{1 - \rho} - 1 \right)
\]

\[
= -a_1^2 W_1 R^{-1} (\beta R^\rho)^{1/\rho} \frac{\rho}{1 - \rho}.
\]  

\[\text{(A.38)}\]

This completes the proof.  

\[\blacksquare\]
Proof of Proposition 4

The optimal portfolio, \( w_{1,0} \), is chosen to maximize the certainty-equivalent \( \mu_0(h_1Z_1) \). From equation (4) it follows that

\[
\mu_0(h_1Z_1) = (E_0(h_1^\alpha Z_1^\alpha))^{1/\alpha}. \tag{A.39}
\]

At date 1, \( h_1 \) can take 2 possible values depending on the state: \( h_1(U) \) when the riskless interest rate is \( R_U \) and the return on the risky asset is 0, and \( h_1(D) \) when the riskless rate is \( R_D \) and the return on the risky asset is \( 2R \). Hence,

\[
(E_0(h_1^\alpha Z_1^\alpha))^{1/\alpha} = \left( E_0(h_1^\alpha ((z_{1,1} - R)w_{1,0} + R)^\alpha) \right)^{1/\alpha} = \left( \frac{1}{2} h_1^\alpha(D)((2R - R)w_{1,0} + R)^\alpha + \frac{1}{2} h_1^\alpha(U)((0 - R)w_{1,0} + R)^\alpha \right)^{1/\alpha} = \left( \frac{1}{2} h_1^\alpha(D)R_1^\alpha(1 + w_{1,0})^\alpha + \frac{1}{2} h_1^\alpha(U)R_1^\alpha(1 - w_{1,0})^\alpha \right)^{1/\alpha} = \left( \frac{1}{2} R \right)^{1/\alpha} \left( h_1^\alpha(D)(1 + w_{1,0})^\alpha + h_1^\alpha(U)(1 - w_{1,0})^\alpha \right)^{1/\alpha}. \tag{A.40}
\]

The expression on the right-hand side of equation (A.40) is maximized with respect to the portfolio weight when the following first order condition is satisfied:

\[
h_1^\alpha(U)(1 - w_{1,0})^\alpha - 1 = h_1^\alpha(D)(1 + w_{1,0})^\alpha - 1 \tag{A.41}
\]

This first-order condition tells us that at the optimum the marginal gain from more investment in the risky asset in the down-state is equal to the marginal cost from holding more of the risky asset in the up-state.

Solving the first-order condition for \( w_{1,0} \) yields

\[
w_{1,0} = \frac{h_1^\alpha(U) - h_1^\alpha(D)}{h_1^\alpha(U) + h_1^\alpha(D)}. \tag{A.42}
\]

From equation (A.10) it follows that \( h_t = a_t^{\frac{\alpha - 1}{\rho}} \). Hence \( h(U) = a(U)^{\frac{\alpha - 1}{\rho}} \) and \( h(D) = a(D)^{\frac{\alpha - 1}{\rho}} \), where

\[
a(U) = \left[ 1 + (\beta R_U^\rho)^{\frac{1}{1 - \rho}} \right]^{-1}, \tag{A.43}
\]
\( a(D) = \left[ 1 + \left( \beta R_D^p \right)^{\frac{1}{\rho}} \right]^{-1}. \) \hspace{1cm} (A.44)

Hence the optimal portfolio, \( w_{1,0} \), can be written as

\[
w_{1,0} = \frac{a_1^{(\alpha-1)}(U) - a_1^{(\alpha-1)}(D)}{a_1^{(\alpha-1)}(U) + a_1^{(\alpha-1)}(D)},
\]

or

\[
w_{1,0} = \frac{1 + (\beta R_D^p)^{\frac{1}{\rho}}}{1 + (\beta R_D^p)^{\frac{1}{\rho}}} - \frac{1 + (\beta R_D^p)^{\frac{1}{\rho}}}{1 + (\beta R_D^p)^{\frac{1}{\rho}}} - \frac{1 + (\beta R_D^p)^{\frac{1}{\rho}}}{1 + (\beta R_D^p)^{\frac{1}{\rho}}}.
\] \hspace{1cm} (A.45)

We now consider the case when there is no intermediate consumption. Our aim is to find the optimal portfolio, \( w_{1,0}^N \), which maximizes the date 0 certainty equivalent, \( \mu_0(U_1) \). To find this certainty equivalent, we need to find date 1 utility, \( U_1 \)

\[
U_1 = f(0, \mu_1(U_2)) = \beta^{1/\rho} \mu_1(U_2),
\]

which depends on date 2 utility

\[
U_2 = (1 - \beta)^{1/\rho} W_2.
\] \hspace{1cm} (A.47)

There is no risky asset available for investment at date 1, so we have the budget constraint

\[
W_2 = W_1 Z_2,
\]

where \( Z_2 = \{ R_D, R_U \} \) depending on the state at date 1. It then follows from (A.48) and the above budget constraint that the date 1 certainty equivalent is given by

\[
\mu_1(U_2) = \mu_1 \left( (1 - \beta)^{1/\rho} W_1 Z_2 \right) = (1 - \beta)^{1/\rho} W_1 Z_2.
\] \hspace{1cm} (A.50)
Substituting the date 1 certainty equivalent (A.50) into (A.47) implies that date 1 utility is given by

\[
U_1 = \beta^{1/\rho}(1 - \beta)^{1/\rho}W_1Z_2. \tag{A.51}
\]

From equation (A.51), we can compute the date 0 certainty equivalent:

\[
\mu_0(U_1) = \beta^{1/\rho}(1 - \beta)^{1/\rho}\mu_0(W_1Z_2). \tag{A.52}
\]

Using the date 1 budget constraint

\[
W_1 = W_0Z_1, \tag{A.53}
\]

the date 0 certainty equivalent in (A.52) is

\[
\mu_0(U_1) = \beta^{1/\rho}(1 - \beta)^{1/\rho}W_0\mu_0(Z_1Z_2). \tag{A.54}
\]

Hence the optimal portfolio is determined by maximizing the date 0 certainty equivalent of the date 1 portfolio return, \(Z_1\), multiplied by the date 1 riskless rate (there is no risky asset available for trading at date 1, so \(Z_2\) is equal to the date 1 riskless rate, which is a date 0 random variable), \(\mu_0(Z_1Z_2)\), i.e.

\[
\mu_0(Z_1Z_2) = \mu_0\left((|z_1 - R|w_{1,0}^N + R) Z_2\right)
\]

\[
= \left[\frac{1}{2} \left((2R - R|w_{1,0}^N + R) R_D\right)^\alpha + \frac{1}{2} \left((0 - R|w_{1,0}^N + R) R_U\right)^\alpha\right]^{1/\alpha} \tag{A.55}
\]

At the maximum the following first-order condition is satisfied:

\[
R_D^\alpha (1 + w_{1,0}^N)^{\alpha-1} = R_U^\alpha (1 - w_{1,0}^N)^{\alpha-1}. \tag{A.55}
\]

Solving (A.55) for \(w_{1,0}^N\) and simplifying gives

\[
w_{1,0}^N = \frac{R_D^{\alpha \alpha} - R_U^{\alpha \alpha}}{R_D^{\alpha \alpha} + R_U^{\alpha \alpha}}, \tag{A.56}
\]

which is the result in the proposition.
We conclude by making the following observation about relative consumption in state \( U \) and \( D \) for the general case with intermediate consumption. Note that relative consumption in the two states is given by

\[
\frac{C_1(D)}{C_1(U)} = \frac{a_1(D) W_1(D)}{a_1(U) W_1(U)} = \frac{a_1(D)(W_0 - C_0)[2Rw_{1,0} + R(1-w_{1,0})]}{a_1(U)(W_0 - C_0)R(1-w_{1,0})} = \frac{a_1(D)(1+w_{1,0})}{a_1(U)(1-w_{1,0})}.
\]

\( (A.57) \)

From (A.45), it follows that

\[
1 + w_{1,0} = \frac{2a_1^{\alpha} (U)}{a_1^{\alpha} (U)+a_1^{\alpha} (D)}, \quad (A.58)
\]

and

\[
1 - w_{1,0} = \frac{2a_1^{\alpha} (D)}{a_1^{\alpha} (U)+a_1^{\alpha} (D)}, \quad (A.59)
\]

Therefore,

\[
\frac{1 + w_{1,0}}{1 - w_{1,0}} = \frac{a_1^{\alpha}(U)}{a_1^{\alpha}(D)}. \quad (A.60)
\]

Substituting (A.60) into (A.57) to eliminate the term containing \( w_{1,0} \) and simplifying gives

\[
\frac{C_1(D)}{C_1(U)} = \frac{1}{a_1} \frac{a_1^{\alpha}(\rho)}{a_1^{\alpha}(\rho)} \left( \frac{D}{U} \right) \left( \frac{1 - \alpha(\rho - 1)}{\alpha(\rho - 1)} \right)^{1 - \frac{\alpha(\rho - 1)}{\rho(\rho - 1)}}. \quad (A.61)
\]

Under expected utility, that is when \( \alpha = \rho \) and the investor is indifferent between the timing of the resolution of uncertainty, the ratio above would equal one. But, when \( \alpha \neq \rho \), then the investor will care about the trade-off between smoothing consumption over time (which is driven by elasticity of intertemporal substitution) versus smoothing consumption across states (which depends on relative risk aversion).

\[\blacksquare\]
Proof of Proposition 5

The demand for the risky asset at date 0 is chosen to maximize the date 0 certainty equivalent of date 1 utility, $\mu_0(h_1Z_1)$. This is equivalent to maximizing the RHS of equation (A.40), when $\alpha \neq 0$. We have already shown that the resulting optimal portfolio is given by (A.42). Because $R_U > R_D$, one can show that $h_1(D) < h_1(U)$. Hence, it follows from (A.42) that $w_{1,0} > 0$ if and only if $\alpha < 0$ and $w_{1,0} < 0$ if and only if $\alpha > 0$.

We now show that if $\alpha = 0$, then $w_{1,0} = 0$. When $\alpha = 0$ maximizing $\mu_0(h_1Z_1)$ is equivalent to maximizing

$$\frac{1}{2} \ln(1 - w_{1,0}) + \frac{1}{2} \ln(1 + w_{1,0}).$$

(A.62)

Solving the resulting first-order condition gives $w_{1,0} = 0$. Thus, an investor with relative risk aversion equal to unity does not hold the risky asset.

For the case with no intermediate consumption, because $R_U > R_D$, it follows from (28) that $w_{1,0}^N > 0$ if and only if $\alpha < 0$, $w_{1,0}^N < 0$ if and only if $\alpha > 0$, and $w_{1,0}^N = 0$ if and only if $\alpha = 0$. ■

B. Analysis of the case where $\rho = 0$ (but $\alpha$ is not restricted)

We know that for $\rho \neq 0$, the optimal portfolio is given by (27). We take the limit of this expression as $\rho \to 0$, but this has to be done carefully. First note that we can write (27) as

$$w_{1,0} = \frac{k(R_D) - k(R_U)}{k(R_U) + k(R_D)},$$

(B.1)

where

$$k(x) = [1 + (\beta x^\rho)^{\frac{1}{1-\rho}}]^{\frac{\alpha (\rho - 1)}{\rho (\alpha - 1)}}.$$

(B.2)

Taking the logarithm of the above expression gives

$$\ln k(x) = \frac{\alpha (\rho - 1)}{\rho (\alpha - 1)} \ln[1 + (\beta x^\rho)^{\frac{1}{1-\rho}}]$$

$$= \frac{\alpha}{1 - \alpha} \left( \frac{1}{\rho} - 1 \right) \ln[1 + (\beta x^\rho)^{\frac{1}{1-\rho}}].$$

(B.3)

From (B.3), we can see that $\ln k(x)$ has an essential singularity at $\rho = 0$. However, the singular term in the power series expansion of $\ln k(x)$ in $\rho$ around 0, will be independent of $x$. Therefore, in the
expression (B.1) for the portfolio weight, \( w_{1,0} \), the singular terms will cancel, so that \( \lim_{\rho \to 0} w_{1,0} \) is well defined. We now show this.

In order to further simplify (B.3), we compute the following series expansion around \( \rho = 0 \):

\[
\ln \left[ 1 + (\beta x^\rho)^{1/\rho} \right] = \ln \left[ 1 + (\beta x^\rho)^{1/\rho} \right] \bigg|_{\rho=0} + \rho \frac{\partial \ln \left[ 1 + (\beta x^\rho)^{1/\rho} \right]}{\partial \rho} \bigg|_{\rho=0} + O(\rho^2)
\]

\[
= \ln(1 + \beta) + \rho \frac{\beta \ln(\beta x)}{1 + \beta} + j(x),
\]

where \( j(x) \) is second order in \( \rho \). Substituting the above expression into (B.3) gives

\[
\ln k(x) = \frac{\alpha}{1 - \alpha} \left( \frac{1}{\rho} - 1 \right) \left[ \ln(1 + \beta) + \rho \frac{\beta \ln(\beta x)}{1 + \beta} + j(x) \right]
\]

\[
= \frac{\alpha}{1 - \alpha} \left( \frac{1}{\rho} - 1 \right) \left[ \ln(1 + \beta) + \rho \frac{\beta \ln(\beta x)}{1 + \beta} \right] + h(x), \quad (B.4)
\]

where \( h(x) \) is first order in \( \rho \). Simplifying further yields

\[
\ln k(x) = \frac{\alpha}{1 - \alpha} \left[ \frac{\ln(1 + \beta)}{\rho} + \frac{\beta \ln(\beta x)}{1 + \beta} - \ln(1 + \beta) - \rho \frac{\beta \ln(\beta x)}{1 + \beta} \right] + h(x)
\]

\[
= \frac{\alpha}{1 - \alpha} \left( \frac{1}{\rho} - 1 \right) \left[ \ln(1 + \beta) + \frac{\beta \ln(\beta x)}{1 + \beta} \right] + g(x)
\]

\[
= \frac{\alpha}{1 - \alpha} \frac{\ln(1 + \beta)}{\rho} + \frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta x)}{1 + \beta} - \frac{\alpha}{1 - \alpha} \ln(1 + \beta) + g(x), \quad (B.5)
\]

where \( g(x) \) is first order in \( \rho \). Therefore,

\[
k(x) = \exp \left[ \frac{\alpha}{1 - \alpha} \frac{\ln(1 + \beta)}{\rho} \right] \exp \left[ \frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta x)}{1 + \beta} \right] \exp \left[ -\frac{\alpha}{1 - \alpha} \ln(1 + \beta) \right] \exp [g(x)] \quad (B.6)
\]

Substituting (B.6) into the expression (B.1) for the portfolio weight, \( w_{1,0} \), gives

\[
w_{1,0} = \frac{e^{\frac{\alpha}{1 - \alpha} \frac{\ln(1 + \beta)}{\rho} - \frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_D)}{1 + \beta} + g(R_D)}}{e^{\frac{\alpha}{1 - \alpha} \frac{\ln(1 + \beta)}{\rho} - \frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_U)}{1 + \beta} + g(R_U)}} - \frac{e^{\frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_U)}{1 + \beta} - \frac{\alpha}{1 - \alpha} \ln(1 + \beta) + g(R_U)}}{e^{\frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_U)}{1 + \beta} - \frac{\alpha}{1 - \alpha} \ln(1 + \beta) + g(R_D)}}
\]

\[
= \frac{e^{\frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_D)}{1 + \beta} + g(R_D)}}{e^{\frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_U)}{1 + \beta} + g(R_U)}} - \frac{e^{\frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_U)}{1 + \beta} + g(R_U)}}{e^{\frac{\alpha}{1 - \alpha} \frac{\beta \ln(\beta R_D)}{1 + \beta} + g(R_D)}} \quad (B.7)
\]
Note that the singular term \( e^{\frac{\alpha \ln(1+\beta)}{\rho}} \) has cancelled out, so \( \lim_{\rho \to 0} w_{1,0} \) is well defined. When taking the limit of the expression on the right-hand side in (B.7), note that because \( g(x) \) is first order in \( \rho \), then

\[
\lim_{\rho \to 0} \exp [g(x)] = e^0 = 1.
\]

Therefore,

\[
\lim_{\rho \to 0} w_{1,0} = \exp \left[ \frac{\alpha}{1-\alpha} \frac{\beta \ln(\beta R_D)}{1+\beta} \right] - \exp \left[ \frac{\alpha}{1-\alpha} \frac{\beta \ln(\beta R_U)}{1+\beta} \right]
\]

\[
\exp \left[ \frac{\alpha}{1-\alpha} \frac{\beta \ln(\beta R_U)}{1+\beta} \right] + \exp \left[ \frac{\alpha}{1-\alpha} \frac{\beta \ln(\beta R_D)}{1+\beta} \right]
\]

\[
= (\beta R_D) \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta} - (\beta R_U) \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta}
\]

\[
(\beta R_U) \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta} + (\beta R_D) \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta}
\]

\[
= \frac{R_D}{R_U} \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta} - R_U \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta}
\]

\[
= \frac{R_U}{R_D} \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta} + R_D \frac{\alpha}{1-\alpha} \frac{\beta}{1+\beta},
\]

(B.8)

which is the expression we wished to obtain. ■
References


Figure 1: Assets available for investment and their payoffs

The figure shows that at $t = 0$ there are two assets that the investor can hold: the single-period riskless asset with a promised return of $R$ and a risky asset with a return $\tilde{R}$ that can either be 0 or $2R$. At $t = 1$, only a single-period riskless asset is available. At $t = 1$, if the realized return on the risky asset is 0 then the single-period riskless asset offers a safe return of $R_U$, and if the realized return on the risky asset is $2R$ then the single-period riskless asset offers a safe return of $R_D$. 
If risky rate realization = 0,
then current single-period riskfree rate = \( R_U \)

\[
\left\{ \begin{array}{l}
\text{Risky rate} = \bar{R} \\
\text{Single-period riskfree rate} = R
\end{array} \right. \quad \bullet
\]

\[
\left\{ \begin{array}{l}
\text{If risky rate realization} = 2R, \\
\text{then current single-period riskfree rate} = R_D
\end{array} \right. \]

Only single-period bond available

Single-period bond and stock available

Final consumption date

\( t = 0 \) \quad \bullet \quad \bullet \quad \bullet \quad \text{time} \quad t = 1 \quad t = 2